Chapter 10

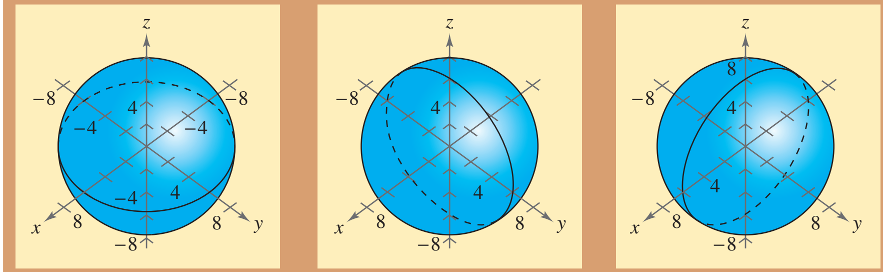
# Analytic Geometry in Three Dimensions



## Selected Applications

Three-dimensional analytic geometry concepts have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

- Crystals, Exercises 75 and 76, pages 748 and 749
- Geography, Exercise 78, page 749
- Tension, Exercises 71 and 72, page 756
- Torque, Exercises 57 and 58, page 763
- Machine Design, Exercise 59, page 772
- Mechanical Design, Exercise 60, page 772




Until now, you have been working mainly with two-dimensional coordinate systems. In Chapter 10, you will learn how to plot points, find distances between points, and represent vectors in three-dimensional coordinate systems. You will also write equations of spheres and graph traces of surfaces in space, as shown above.

Arnold Fisher/Photo Researchers, Inc.

The three-dimensional coordinate system is used in chemistry to help understand the structures of crystals. For example, isometric crystals are shaped like cubes.

# 10.1 The Three-Dimensional Coordinate System

## The Three-Dimensional Coordinate System

Recall that the Cartesian plane is determined by two perpendicular number lines called the $x$-axis and the $y$-axis. These axes, together with their point of intersection (the origin), allow you to develop a two-dimensional coordinate system for identifying points in a plane. To identify a point in space, you must introduce a third dimension to the model. The geometry of this three-dimensional model is called **solid analytic geometry.**

You can construct a **three-dimensional coordinate system** by passing a $z$-axis perpendicular to both the $x$- and $y$-axes at the origin. Figure 10.1 shows the positive portion of each coordinate axis. Taken as pairs, the axes determine three **coordinate planes:** the ***xy*-plane,** the ***xz*-plane,** and the ***yz*-plane.** These three coordinate planes separate the three-dimensional coordinate system into eight **octants.** The first octant is the one in which all three coordinates are positive. In this three-dimensional system, a point $P$ in space is determined by an ordered triple $(x, y, z)$, where $x$, $y$, and $z$ are as follows.

$x =$ directed distance from $yz$-plane to $P$

$y =$ directed distance from $xz$-plane to $P$

$z =$ directed distance from $xy$-plane to $P$

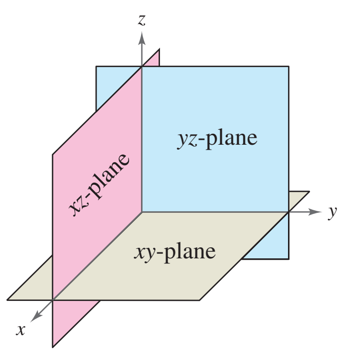



Figure 10.1

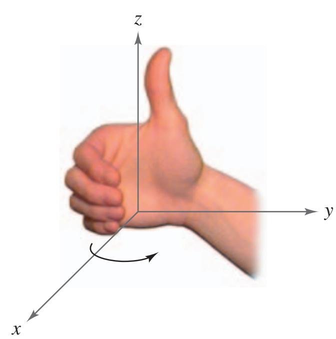
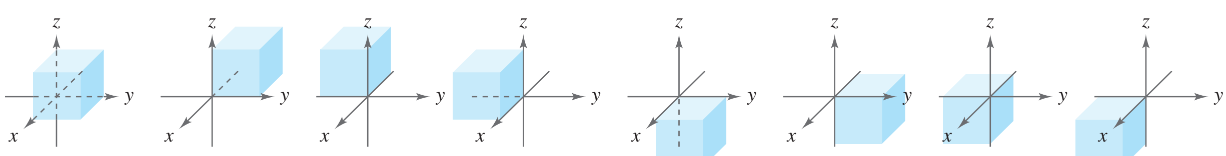



Figure 10.2

A three-dimensional coordinate system can have either a **left-handed** or a **right-handed** orientation. In this text, you will work exclusively with right-handed systems, as illustrated in Figure 10.2. In a right-handed system, Octants II, III, and IV are found by rotating counterclockwise around the positive $z$-axis. Octant V is vertically below Octant I. Octants VI, VII, and VIII are then found by rotating counterclockwise around the negative $z$-axis. See Figure 10.3.




Octant I Octant II Octant III Octant IV Octant V Octant VI Octant VII Octant VIII

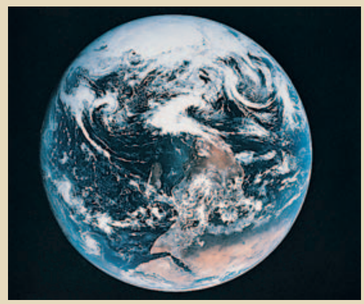
Figure 10.3

### What you should learn

- Plot points in the three-dimensional coordinate system.
- Find distances between points in space and find midpoints of line segments joining points in space.
- Write equations of spheres in standard form and find traces of surfaces in space.

### Why you should learn it

The three-dimensional coordinate system can be used to graph equations that model surfaces in space, such as the spherical shape of Earth, as shown in Exercise 78 on page 749.



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**Prerequisite Skills**

To review the Cartesian Plane, see Appendix B.1.

### Example 1 Plotting Points in Space

Plot each point in space.

**a.** $(2, -3, 3)$ **b.** $(-2, 6, 2)$ **c.** $(1, 4, 0)$ **d.** $(2, 2, -3)$

#### Solution

To plot the point $(2, -3, 3)$, notice that $x = 2$, $y = -3$, and $z = 3$. To help visualize the point, locate the point $(2, -3)$ in the $xy$-plane (denoted by a cross in Figure 10.4). The point $(2, -3, 3)$ lies three units above the cross. The other three points are also shown in Figure 10.4.

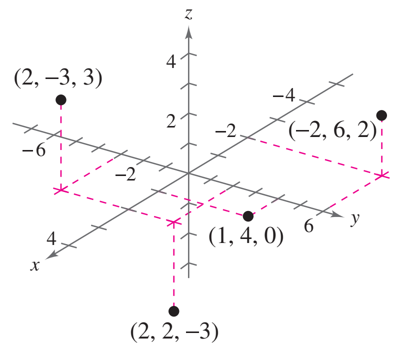



Figure 10.4

CHECKPOINT Now try Exercise 7.

## The Distance and Midpoint Formulas

Many of the formulas established for the two-dimensional coordinate system can be extended to three dimensions. For example, to find the distance between two points in space, you can use the Pythagorean Theorem twice, as shown in Figure 10.5.

> **Distance Formula in Space**
>
> The distance between the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ given by the **Distance Formula in Space** is
>
> $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

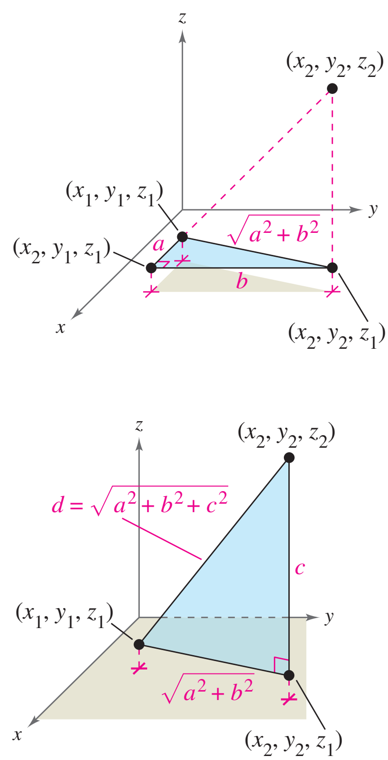



Figure 10.5

### Example 2 Finding the Distance Between Two Points in Space

Find the distance between $(0, 1, 3)$ and $(1, 4, -2)$.

#### Solution

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} && \text{Distance Formula in Space} \\ &= \sqrt{(1 - 0)^2 + (4 - 1)^2 + (-2 - 3)^2} && \text{Substitute.} \\ &= \sqrt{1 + 9 + 25} = \sqrt{35} && \text{Simplify.} \end{aligned}$$

CHECKPOINT Now try Exercise 21.

Notice the similarity between the Distance Formulas in the plane and in space. The Midpoint Formulas in the plane and in space are also similar.

> **Midpoint Formula in Space**
>
> The midpoint of the line segment joining the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ given by the **Midpoint Formula in Space** is
>
> $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

### Example 3 Using the Midpoint Formula in Space

Find the midpoint of the line segment joining $(5, -2, 3)$ and $(0, 4, 4)$.

#### Solution

Using the Midpoint Formula in Space, the midpoint is

$$\left(\frac{5+0}{2}, \frac{-2+4}{2}, \frac{3+4}{2}\right) = \left(\frac{5}{2}, 1, \frac{7}{2}\right)$$

as shown in Figure 10.6.

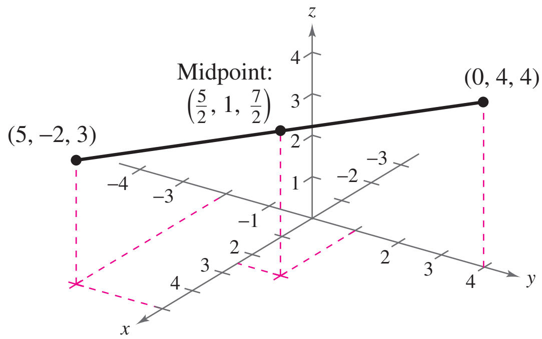



**Figure 10.6**

CHECKPOINT Now try Exercise 35.

## The Equation of a Sphere

A **sphere** with center $(h, k, j)$ and radius $r$ is defined as the set of all points $(x, y, z)$ such that the distance between $(x, y, z)$ and $(h, k, j)$ is $r$, as shown in Figure 10.7. Using the Distance Formula, this condition can be written as

$$\sqrt{(x-h)^2 + (y-k)^2 + (z-j)^2} = r.$$

By squaring each side of this equation, you obtain the standard equation of a sphere.

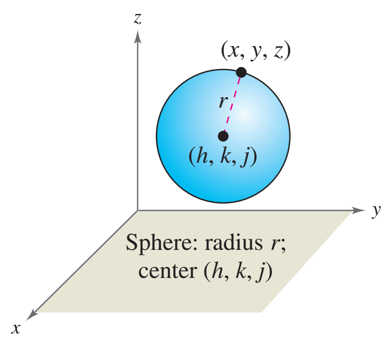



**Figure 10.7**

**Standard Equation of a Sphere**

The **standard equation of a sphere** with center $(h, k, j)$ and radius $r$ is given by

$$(x-h)^2 + (y-k)^2 + (z-j)^2 = r^2.$$

Notice the similarity of this formula to the equation of a circle in the plane.

$$(x-h)^2 + (y-k)^2 + (z-j)^2 = r^2 \quad \text{Equation of sphere in space}$$

$$(x-h)^2 + (y-k)^2 = r^2 \quad \text{Equation of circle in the plane}$$

As is true with the equation of a circle, the equation of a sphere is simplified when the center lies at the origin. In this case, the equation is

$$x^2 + y^2 + z^2 = r^2. \quad \text{Sphere with center at origin}$$

## Example 4 Finding the Equation of a Sphere

Find the standard equation of the sphere with center $(2, 4, 3)$ and radius 3. Does this sphere intersect the $xy$-plane?

### Solution

$$(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2 \qquad \text{Standard equation}$$

$$(x - 2)^2 + (y - 4)^2 + (z - 3)^2 = 3^2 \qquad \text{Substitute.}$$

From the graph shown in Figure 10.8, you can see that the center of the sphere lies three units above the $xy$-plane. Because the sphere has a radius of 3, you can conclude that it does intersect the $xy$-plane—at the point $(2, 4, 0)$.

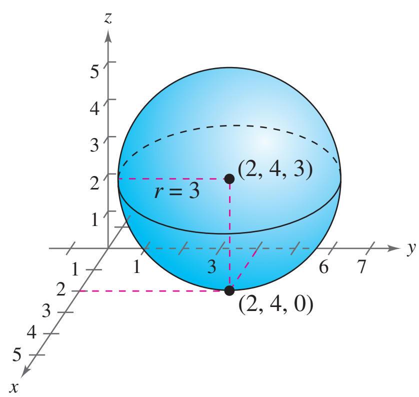



**Figure 10.8**

CHECKPOINT Now try Exercise 45.

**Exploration**

Find the equation of the sphere that has the points $(3, -2, 6)$ and $(-1, 4, 2)$ as endpoints of a diameter. Explain how this problem gives you a chance to use all three formulas discussed so far in this section: the Distance Formula in Space, the Midpoint Formula in Space, and the standard equation of a sphere.

## Example 5 Finding the Center and Radius of a Sphere

Find the center and radius of the sphere given by

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 8 = 0.$$

### Solution

To obtain the standard equation of this sphere, complete the square as follows.

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 8 = 0$$

$$(x^2 - 2x + \quad) + (y^2 + 4y + \quad) + (z^2 - 6z + \quad) = -8$$

$$(x^2 - 2x + 1) + (y^2 + 4y + 4) + (z^2 - 6z + 9) = -8 + 1 + 4 + 9$$

$$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = \left(\sqrt{6}\right)^2$$

So, the center of the sphere is $(1, -2, 3)$, and its radius is $\sqrt{6}$. See Figure 10.9.

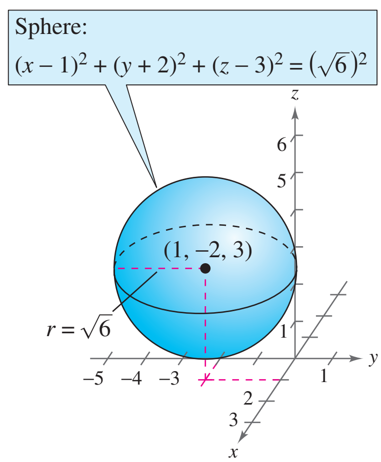



**Figure 10.9**

CHECKPOINT Now try Exercise 55.

Note in Example 5 that the points satisfying the equation of the sphere are "surface points," not "interior points." In general, the collection of points satisfying an equation involving $x$, $y$, and $z$ is called a **surface in space.**

Finding the intersection of a surface with one of the three coordinate planes (or with a plane parallel to one of the three coordinate planes) helps one visualize the surface. Such an intersection is called a **trace** of the surface. For example, the $xy$-trace of a surface consists of all points that are common to both the surface *and* the $xy$-plane. Similarly, the $xz$-trace of a surface consists of all points that are common to both the surface and the $xz$-plane.

## Example 6 Finding a Trace of a Surface

Sketch the $xy$-trace of the sphere given by $(x - 3)^2 + (y - 2)^2 + (z + 4)^2 = 5^2$.

### Solution

To find the $xy$-trace of this surface, use the fact that every point in the $xy$-plane has a $z$-coordinate of zero. This means that if you substitute $z = 0$ into the original equation, the resulting equation will represent the intersection of the surface with the $xy$-plane.

$$(x - 3)^2 + (y - 2)^2 + (z + 4)^2 = 5^2 \qquad \text{Write original equation.}$$

$$(x - 3)^2 + (y - 2)^2 + (0 + 4)^2 = 5^2 \qquad \text{Substitute 0 for } z.$$

$$(x - 3)^2 + (y - 2)^2 + 16 = 25 \qquad \text{Simplify.}$$

$$(x - 3)^2 + (y - 2)^2 = 9 \qquad \text{Subtract 16 from each side.}$$

$$(x - 3)^2 + (y - 2)^2 = 3^2 \qquad \text{Equation of circle}$$

From this form, you can see that the $xy$-trace is a circle of radius 3, as shown in Figure 10.10.

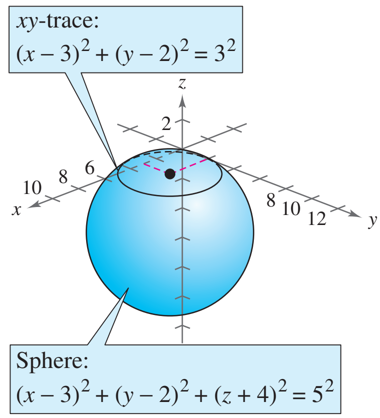



**Figure 10.10**

CHECKPOINT Now try Exercise 65.

**TECHNOLOGY TIP** Most three-dimensional graphing utilities and computer algebra systems represent surfaces by sketching several traces of the surface. The traces are usually taken in equally spaced parallel planes. To graph an equation involving $x$, $y$, and $z$ with a three-dimensional "function grapher," you must first set the graphing mode to *three-dimensional* and solve the equation for $z$. After entering the equation, you need to specify a rectangular viewing cube (the three-dimensional analog of a viewing window). For instance, to graph the top half of the sphere from Example 6, solve the equation for $z$ to obtain the solutions $z = -4 \pm \sqrt{25 - (x - 3)^2 - (y - 2)^2}$. The equation $z = -4 + \sqrt{25 - (x - 3)^2 - (y - 2)^2}$ represents the top half of the sphere. Enter this equation, as shown in Figure 10.11. Next, use the viewing cube shown in Figure 10.12. Finally, you can display the graph, as shown in Figure 10.13.

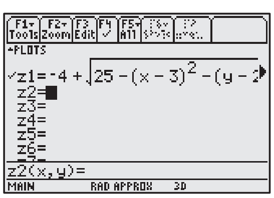



**Figure 10.11**

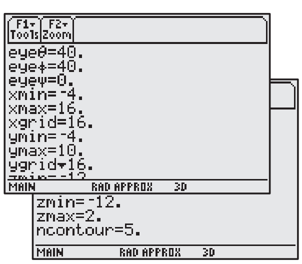



**Figure 10.12**

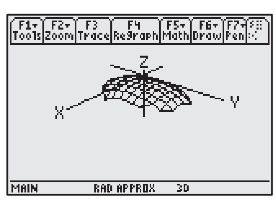



**Figure 10.13**

## 10.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

### *Vocabulary Check*

**Fill in the blanks.**

**1.** A _______ coordinate system can be formed by passing a $z$-axis perpendicular to both the $x$-axis and the $y$-axis at the origin.

**2.** The three coordinate planes of a three-dimensional coordinate system are the _______ , the _______ , and the _______ .

**3.** The coordinate planes of a three-dimensional coordinate system separate the coordinate system into eight _______ .

**4.** The distance between the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ can be found using the _______ in Space.

**5.** The midpoint of the line segment joining the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ given by the Midpoint Formula in Space is _______.

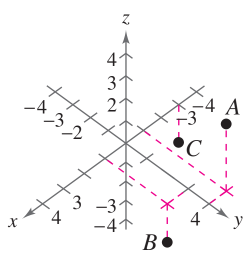
**6.** A _______ is the set of all points $(x, y, z)$ such that the distance between $(x, y, z)$ and a fixed point $(h, k, j)$ is $r$.

**7.** A _______ in _______ is the collection of points satisfying an equation involving $x$, $y$, and $z$.

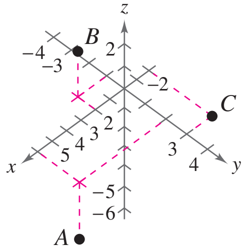
**8.** The intersection of a surface with one of the three coordinate planes is called a _______ of the surface.

**In Exercises 1–4, approximate the coordinates of the points.**

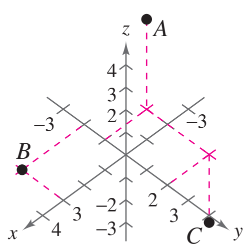
**1.**



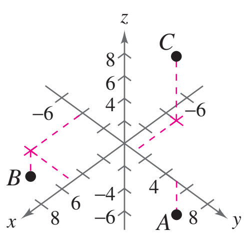

**2.**




**3.**




**4.**




**In Exercises 5–10, plot each point in the same three-dimensional coordinate system.**

**5.** (a) $(2, 1, 3)$
(b) $(-1, 2, 1)$

**6.** (a) $(3, 0, 0)$
(b) $(-3, -2, -1)$

**7.** (a) $(3, -1, 0)$
(b) $(-4, 2, 2)$

**8.** (a) $(0, 4, -3)$
(b) $(4, 0, 4)$

**9.** (a) $(3, -2, 5)$
(b) $\left(\frac{3}{2}, 4, -2\right)$

**10.** (a) $(5, -2, 2)$
(b) $(5, -2, -2)$

**In Exercises 11–14, find the coordinates of the point.**

**11.** The point is located three units behind the $yz$-plane, three units to the right of the $xz$-plane, and four units above the $xy$-plane.

**12.** The point is located six units in front of the $yz$-plane, one unit to the left of the $xz$-plane, and one unit below the $xy$-plane.

**13.** The point is located on the $x$-axis, 10 units in front of the $yz$-plane.

**14.** The point is located in the $yz$-plane, two units to the right of the $xz$-plane, and eight units above the $xy$-plane.

**In Exercises 15–20, determine the octant(s) in which $(x, y, z)$ is located so that the condition(s) is (are) satisfied.**

**15.** $x > 0, y < 0, z > 0$

**16.** $x < 0, y > 0, z < 0$

**17.** $z > 0$

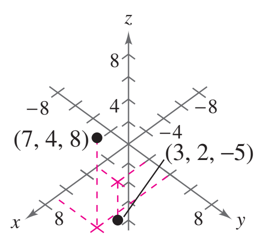
**18.** $y < 0$

**19.** $xy < 0$

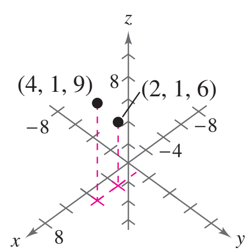
**20.** $yz > 0$

**In Exercises 21–26, find the distance between the points.**

**21.**




**22.**




**23.** $(-1, 4, -2), (6, 0, -9)$

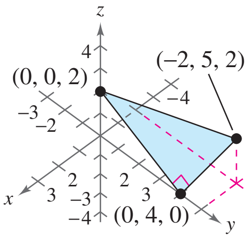
**24.** $(1, 1, -7), (-2, -3, -7)$

**25.** $(0, -3, 0), (1, 0, -10)$

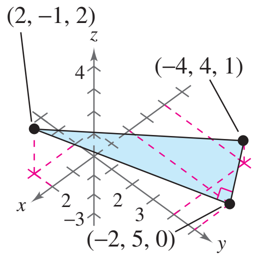
**26.** $(2, -4, 0), (0, 6, -3)$

**In Exercises 27–30, find the lengths of the sides of the right triangle. Show that these lengths satisfy the Pythagorean Theorem.**

**27.**




**28.**




**29.** $(0, 0, 0), (2, 2, 1), (2, -4, 4)$

**30.** $(1, 0, 1), (1, 3, 1), (1, 0, 3)$

**In Exercises 31–34, find the lengths of the sides of the triangle with the indicated vertices, and determine whether the triangle is a right triangle, an isosceles triangle, or neither.**

**31.** $(1, -3, -2), (5, -1, 2), (-1, 1, 2)$

**32.** $(5, 3, 4), (7, 1, 3), (3, 5, 3)$

**33.** $(4, -1, -2), (8, 1, 2), (2, 3, 2)$

**34.** $(1, -2, -1), (3, 0, 0), (3, -6, 3)$

**In Exercises 35–40, find the midpoint of the line segment joining the points.**

**35.** $(3, -6, 10), (-3, 4, 4)$

**36.** $(-1, 5, -3), (3, 7, -1)$

**37.** $(6, -2, 5), (-4, 2, 6)$

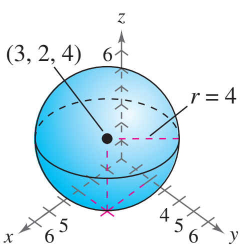
**38.** $(-3, 5, 5), (-6, 4, 8)$

**39.** $(-2, 8, 10), (7, -4, 2)$

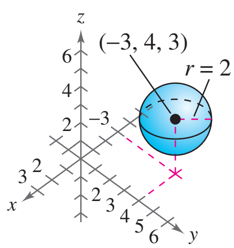
**40.** $(9, -5, 1), (9, -2, -4)$

**In Exercises 41–50, find the standard form of the equation of the sphere with the given characteristic.**

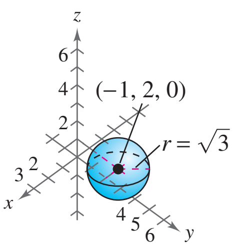
**41.**



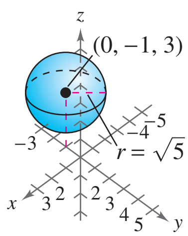

**42.**




**43.**




**44.**




**45.** Center: $(0, 4, 3)$; radius: 3

**46.** Center: $(2, -1, 8)$; radius: 6

**47.** Center: $(-3, 7, 5)$; diameter: 10

**48.** Center: $(0, 5, -9)$; diameter: 8

**49.** Endpoints of a diameter: $(3, 0, 0), (0, 0, 6)$

**50.** Endpoints of a diameter: $(2, -2, 2), (-1, 4, 6)$

**In Exercises 51–64, find the center and radius of the sphere.**

**51.** $x^2 + y^2 + z^2 - 5x = 0$

**52.** $x^2 + y^2 + z^2 - 8y = 0$

**53.** $x^2 + y^2 + z^2 - 4x + 2y = 0$

**54.** $x^2 + y^2 + z^2 - x - y - z = 0$

**55.** $x^2 + y^2 + z^2 - 4x + 2y - 6z + 10 = 0$

**56.** $x^2 + y^2 + z^2 - 6x + 4y + 9 = 0$

**57.** $x^2 + y^2 + z^2 + 4x - 8z + 19 = 0$

**58.** $x^2 + y^2 + z^2 - 8y - 6z + 13 = 0$

**59.** $9x^2 + 9y^2 + 9z^2 - 18x - 6y - 72z + 73 = 0$

**60.** $2x^2 + 2y^2 + 2z^2 - 2x - 6y - 4z + 5 = 0$

**61.** $4x^2 + 4y^2 + 4z^2 - 8x + 16y - 1 = 0$

**62.** $9x^2 + 9y^2 + 9z^2 - 18x + 36y + 54 - 126 = 0$

**63.** $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$

**64.** $4x^2 + 4y^2 + 4z^2 - 4x - 32y + 8z + 33 = 0$

**In Exercises 65–70, sketch the graph of the equation and sketch the specified trace.**

**65.** $(x - 1)^2 + y^2 + z^2 = 36$; $xz$-trace

**66.** $x^2 + (y + 3)^2 + z^2 = 25$; $yz$-trace

**67.** $(x + 2)^2 + (y - 3)^2 + z^2 = 9$; $yz$-trace

**68.** $x^2 + (y - 1)^2 + (z + 1)^2 = 4$; $xy$-trace

**69.** $x^2 + y^2 + z^2 - 2x - 4z + 1 = 0$; $yz$-trace

**70.** $x^2 + y^2 + z^2 - 4y - 6z - 12 = 0$; $xz$-trace

**In Exercises 71–74, use a three-dimensional graphing utility to graph the sphere.**

**71.** $x^2 + y^2 + z^2 - 6x - 8y - 10z + 46 = 0$

**72.** $x^2 + y^2 + z^2 + 6y - 8z + 21 = 0$

**73.** $4x^2 + 4y^2 + 4z^2 - 8x - 16y + 8z - 25 = 0$

**74.** $9x^2 + 9y^2 + 9z^2 + 18x - 18y + 36z + 35 = 0$

**75.** ***Crystals*** Crystals are classified according to their symmetry. Crystals shaped like cubes are classified as isometric. The vertices of an isometric crystal mapped onto a three-dimensional coordinate system are shown in the figure on the next page. Determine $(x, y, z)$.

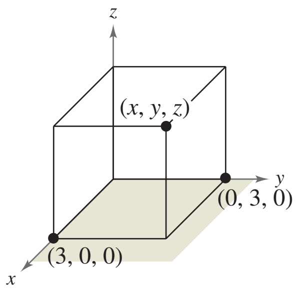



Figure for 75

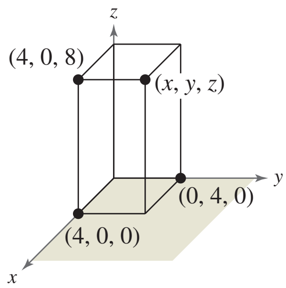



Figure for 76

**76.** ***Crystals*** Crystals shaped like rectangular prisms are classified as tetragonal. The vertices of a tetragonal crystal mapped onto a three-dimensional coordinate system are shown in the figure. Determine $(x, y, z)$.

**77.** ***Architecture*** A spherical building has a diameter of 165 feet. The center of the building is placed at the origin of a three-dimensional coordinate system. What is the equation of the sphere?

**78.** ***Geography*** Assume that Earth is a sphere with a radius of 3963 miles. The center of Earth is placed at the origin of a three-dimensional coordinate system.

(a) What is the equation of the sphere?

(b) Lines of longitude that run north-south could be represented by what trace(s)? What shape would each of these traces form?

(c) Lines of latitude that run east-west could be represented by what trace(s)? What shape would each of these traces form?

## Synthesis

***True or False?*** **In Exercises 79 and 80, determine whether the statement is true or false. Justify your answer.**

**79.** In the ordered triple $(x, y, z)$ that represents point $P$ in space, $x$ is the directed distance from the $xy$-plane to $P$.

**80.** The surface consisting of all points $(x, y, z)$ in space that are the same distance $r$ from the point $(h, k, j)$ has a circle as its $xy$-trace.

**81.** ***Think About It*** What is the $z$-coordinate of any point in the $xy$-plane? What is the $y$-coordinate of any point in the $xz$-plane? What is the $x$-coordinate of any point in the $yz$-plane?

**82.** ***Writing*** In two-dimensional coordinate geometry, the graph of the equation $ax + by + c = 0$ is a line. In three-dimensional coordinate geometry, what is the graph of the equation $ax + by + cz = 0$? Is it a line? Explain your reasoning.

**83.** A sphere intersects the $yz$-plane. Describe the trace.

**84.** A plane intersects the $xy$-plane. Describe the trace.

**85.** A line segment has $(x_1, y_1, z_1)$ as one endpoint and $(x_m, y_m, z_m)$ as its midpoint. Find the other endpoint $(x_2, y_2, z_2)$ of the line segment in terms of $x_1, y_1, z_1, x_m, y_m$, and $z_m$.

**86.** Use the result of Exercise 85 to find the coordinates of one endpoint of a line segment if the coordinates of the other endpoint and the midpoint are $(3, 0, 2)$ and $(5, 8, 7)$, respectively.

## Skills Review

**In Exercises 87–92, solve the quadratic equation by completing the square.**

**87.** $v^2 + 3v - 2 = 0$ **88.** $z^2 - 7z - 19 = 0$

**89.** $x^2 - 5x + 5 = 0$ **90.** $x^2 + 3x - 1 = 0$

**91.** $4y^2 + 4y - 9 = 0$ **92.** $2x^2 + 5x - 8 = 0$

**In Exercises 93–96, find the magnitude and direction angle of the vector v.**

**93.** $\mathbf{v} = 3\mathbf{i} - 3\mathbf{j}$ **94.** $\mathbf{v} = -\mathbf{i} + 2\mathbf{j}$

**95.** $\mathbf{v} = 4\mathbf{i} + 5\mathbf{j}$ **96.** $\mathbf{v} = 10\mathbf{i} - 7\mathbf{j}$

**In Exercises 97 and 98, find the dot product of u and v.**

**97.** $\mathbf{u} = \langle -4, 1 \rangle$, $\mathbf{v} = \langle 3, 5 \rangle$ **98.** $\mathbf{u} = \langle -1, 0 \rangle$, $\mathbf{v} = \langle -2, -6 \rangle$

**In Exercises 99–102, write the first five terms of the sequence beginning with the given term. Then calculate the first and second differences of the sequence. State whether the sequence has a linear model, a quadratic model, or neither.**

**99.** $a_0 = 1$, $a_n = a_{n-1} + n^2$ **100.** $a_0 = 0$, $a_n = a_{n-1} - 1$

**101.** $a_1 = -1$, $a_n = a_{n-1} + 3$ **102.** $a_1 = 4$, $a_n = a_{n-1} - 2n$

**In Exercises 103–110, find the standard form of the equation of the conic with the given characteristics.**

**103.** Circle: center: $(-5, 1)$; radius: 7

**104.** Circle: center: $(3, -6)$; radius: 9

**105.** Parabola: vertex: $(4, 1)$; focus: $(1, 1)$

**106.** Parabola: vertex: $(-2, 5)$; focus: $(-2, 0)$

**107.** Ellipse: vertices: $(0, 3)$, $(6, 3)$; minor axis of length 4

**108.** Ellipse: foci: $(0, 0)$, $(0, 6)$; major axis of length 9

**109.** Hyperbola: vertices: $(4, 0)$, $(8, 0)$; foci: $(0, 0)$, $(12, 0)$

**110.** Hyperbola: vertices: $(3, 1)$, $(3, 9)$; foci: $(3, 0)$, $(3, 10)$

# 10.2 Vectors in Space

## Vectors in Space

Physical forces and velocities are not confined to the plane, so it is natural to extend the concept of vectors from two-dimensional space to three-dimensional space. In space, vectors are denoted by ordered triples

$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Component form

The **zero vector** is denoted by $\mathbf{0} = \langle 0, 0, 0 \rangle$. Using the unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ in the direction of the positive $z$-axis, the **standard unit vector notation** for $\mathbf{v}$ is

$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ Unit vector form

as shown in Figure 10.14. If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, as shown in Figure 10.15, the **component form** of $\mathbf{v}$ is produced by subtracting the coordinates of the initial point from the coordinates of the terminal point

$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$.

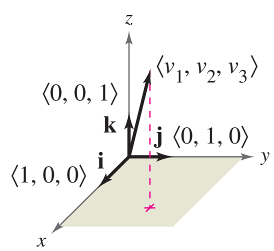



**Figure 10.14**

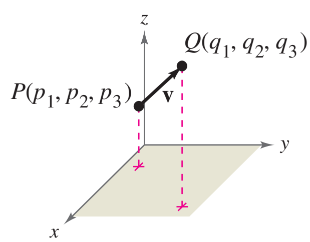



**Figure 10.15**

### What you should learn

- Find the component forms of, the unit vectors in the same direction of, the magnitudes of, the dot products of, and the angles between vectors in space.
- Determine whether vectors in space are parallel or orthogonal.
- Use vectors in space to solve real-life problems.

### Why you should learn it

Vectors in space can be used to represent many physical forces, such as tension in the wires used to support auditorium lights, as shown in Exercise 71 on page 756.

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### Vectors in Space

1. Two vectors are **equal** if and only if their corresponding components are equal.
2. The **magnitude** (or **length**) of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$.
3. A **unit vector u** in the direction of $\mathbf{v}$ is $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$, $\mathbf{v} \neq \mathbf{0}$.
4. The **sum** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

   $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$. Vector addition
5. The **scalar multiple** of the real number $c$ and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is

   $c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$. Scalar multiplication
6. The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

   $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$. Dot product

**Prerequisite Skills**

To review definitions of vectors in the plane, see Section 6.3.

## Example 1 Finding the Component Form of a Vector

Find the component form and magnitude of the vector **v** having initial point (3, 4, 2) and terminal point (3, 6, 4). Then find a unit vector in the direction of **v**.

### Solution

The component form of **v** is

$$\mathbf{v} = \langle 3 - 3, 6 - 4, 4 - 2 \rangle = \langle 0, 2, 2 \rangle$$

which implies that its magnitude is

$$\|\mathbf{v}\| = \sqrt{0^2 + 2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

The unit vector in the direction of **v** is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{2\sqrt{2}}\langle 0, 2, 2 \rangle = \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle.$$

CHECKPOINT Now try Exercise 5.

## Example 2 Finding the Dot Product of Two Vectors

Find the dot product of $\langle 4, 0, 1 \rangle$ and $\langle -1, 3, 2 \rangle$.

### Solution

$$\begin{aligned} \langle 4, 0, 1 \rangle \cdot \langle -1, 3, 2 \rangle &= 4(-1) + 0(3) + 1(2) \\ &= -4 + 0 + 2 = -2 \end{aligned}$$

Note that the dot product of two vectors is a real number, not a vector.

CHECKPOINT Now try Exercise 39.

**TECHNOLOGY TIP**

Some graphing utilities have the capability to perform vector operations, such as the dot product. Consult the user's guide for your graphing utility for specific instructions.

As was discussed in Section 6.4, the **angle between two nonzero vectors** is the angle $\theta$, $0 \le \theta \le \pi$, between their respective standard position vectors, as shown in Figure 10.16. This angle can be found using the dot product. (Note that the angle between the zero vector and another vector is not defined.)

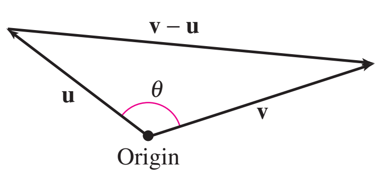


**Figure 10.16**

**Angle Between Two Vectors**

If $\theta$ is the angle between two nonzero vectors **u** and **v**, then $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$.

If the dot product of two nonzero vectors is zero, the angle between the vectors is 90°. Such vectors are called **orthogonal.** For instance, the standard unit vectors **i**, **j**, and **k** are orthogonal to each other.

### Example 3 Finding the Angle Between Two Vectors

Find the angle between $\mathbf{u} = \langle 1, 0, 2 \rangle$ and $\mathbf{v} = \langle 3, 1, 0 \rangle$.

#### Solution

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{\langle 1, 0, 2 \rangle \cdot \langle 3, 1, 0 \rangle}{\|\langle 1, 0, 2 \rangle\| \, \|\langle 3, 1, 0 \rangle\|} = \frac{3}{\sqrt{50}}$$

This implies that the angle between the two vectors is

$$\theta = \arccos \frac{3}{\sqrt{50}} \approx 64.9°$$

as shown in Figure 10.17.

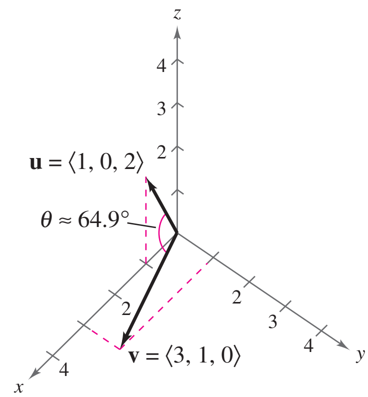

**Figure 10.17**

CHECKPOINT Now try Exercise 43.

## Parallel Vectors

Recall from the definition of scalar multiplication that positive scalar multiples of a nonzero vector $\mathbf{v}$ have the same direction as $\mathbf{v}$, whereas negative multiples have the direction opposite that of $\mathbf{v}$. In general, two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if there is some scalar $c$ such that $\mathbf{u} = c\mathbf{v}$. For example, in Figure 10.18, the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are parallel because $\mathbf{u} = 2\mathbf{v}$ and $\mathbf{w} = -\mathbf{v}$.

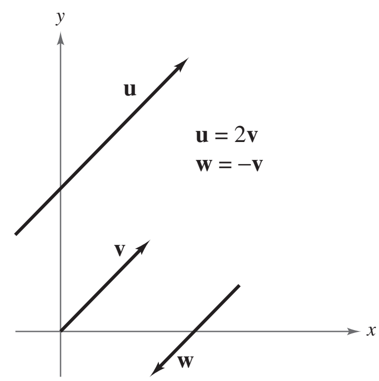

**Figure 10.18**

### Example 4 Parallel Vectors

Vector $\mathbf{w}$ has initial point $(1, -2, 0)$ and terminal point $(3, 2, 1)$. Which of the following vectors is parallel to $\mathbf{w}$?

**a.** $\mathbf{u} = \langle 4, 8, 2 \rangle$ **b.** $\mathbf{v} = \langle 4, 8, 4 \rangle$

#### Solution

Begin by writing $\mathbf{w}$ in component form.

$$\mathbf{w} = \langle 3 - 1, 2 - (-2), 1 - 0 \rangle = \langle 2, 4, 1 \rangle$$

**a.** Because

$$\begin{aligned} \mathbf{u} &= \langle 4, 8, 2 \rangle \\ &= 2\langle 2, 4, 1 \rangle \\ &= 2\mathbf{w} \end{aligned}$$

you can conclude that $\mathbf{u}$ *is* parallel to $\mathbf{w}$.

**b.** In this case, you need to find a scalar $c$ such that

$$\langle 4, 8, 4 \rangle = c\langle 2, 4, 1 \rangle.$$

However, equating corresponding components produces $c = 2$ for the first two components and $c = 4$ for the third. So, the equation has no solution, and the vectors $\mathbf{v}$ and $\mathbf{w}$ are *not* parallel.

CHECKPOINT Now try Exercise 47.

You can use vectors to determine whether three points are collinear (lie on the same line). The points $P$, $Q$, and $R$ are **collinear** if and only if the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel.

### Example 5 Using Vectors to Determine Collinear Points

Determine whether the points $P(2, -1, 4)$, $Q(5, 4, 6)$, and $R(-4, -11, 0)$ are collinear.

#### Solution

The component forms of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{PQ} = \langle 5 - 2, 4 - (-1), 6 - 4 \rangle = \langle 3, 5, 2 \rangle$$

and

$$\overrightarrow{PR} = \langle -4 - 2, -11 - (-1), 0 - 4 \rangle = \langle -6, -10, -4 \rangle.$$

Because $\overrightarrow{PR} = -2\overrightarrow{PQ}$, you can conclude that they are parallel. Therefore, the points $P$, $Q$, and $R$ lie on the same line, as shown in Figure 10.19.

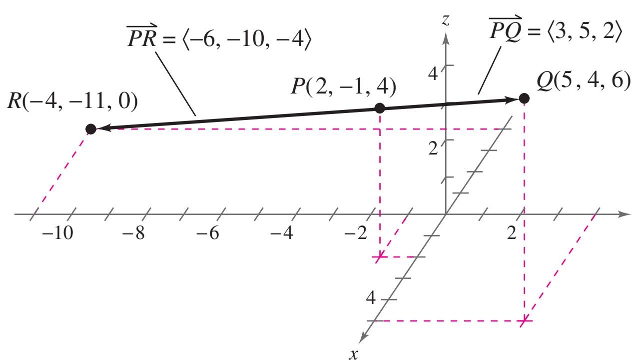


**Figure 10.19**

CHECKPOINT Now try Exercise 57.

### Example 6 Finding the Terminal Point of a Vector

The initial point of the vector $\mathbf{v} = \langle 4, 2, -1 \rangle$ is $P(3, -1, 6)$. What is the terminal point of this vector?

#### Solution

Using the component form of the vector whose initial point is $P(3, -1, 6)$ and whose terminal point is $Q(q_1, q_2, q_3)$, you can write

$$\begin{aligned}\overrightarrow{PQ} &= \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle \\ &= \langle q_1 - 3, q_2 + 1, q_3 - 6 \rangle = \langle 4, 2, -1 \rangle.\end{aligned}$$

This implies that $q_1 - 3 = 4$, $q_2 + 1 = 2$, and $q_3 - 6 = -1$. The solutions of these three equations are $q_1 = 7$, $q_2 = 1$, and $q_3 = 5$. So, the terminal point is $Q(7, 1, 5)$.

CHECKPOINT Now try Exercise 63.

## Application

In Section 6.3, you saw how to use vectors to solve an equilibrium problem in a plane. The next example shows how to use vectors to solve an equilibrium problem in space.

### Example 7 Solving an Equilibrium Problem

A weight of 480 pounds is supported by three ropes. As shown in Figure 10.20, the weight is located at $S(0, 2, -1)$. The ropes are tied to the points $P(2, 0, 0)$, $Q(0, 4, 0)$, and $R(-2, 0, 0)$. Find the force (or tension) on each rope.

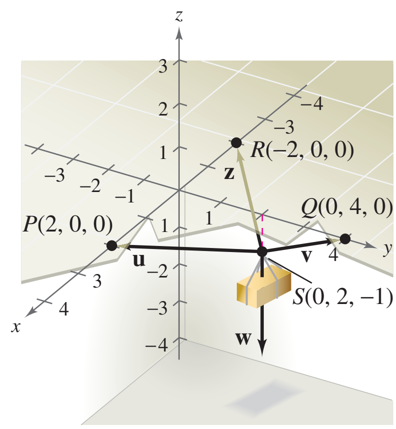


Figure 10.20

#### Solution

The (downward) force of the weight is represented by the vector

$$\mathbf{w} = \langle 0, 0, -480 \rangle.$$

The force vectors corresponding to the ropes are as follows.

$$\mathbf{u} = \|\mathbf{u}\| \frac{\overrightarrow{SP}}{\|\overrightarrow{SP}\|} = \|\mathbf{u}\| \frac{\langle 2-0, 0-2, 0-(-1)\rangle}{3} = \|\mathbf{u}\| \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

$$\mathbf{v} = \|\mathbf{v}\| \frac{\overrightarrow{SQ}}{\|\overrightarrow{SQ}\|} = \|\mathbf{v}\| \frac{\langle 0-0, 4-2, 0-(-1)\rangle}{\sqrt{5}} = \|\mathbf{v}\| \left\langle 0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$\mathbf{z} = \|\mathbf{z}\| \frac{\overrightarrow{SR}}{\|\overrightarrow{SR}\|} = \|\mathbf{z}\| \frac{\langle -2-0, 0-2, 0-(-1)\rangle}{3} = \|\mathbf{z}\| \left\langle -\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

For the system to be in equilibrium, it must be true that

$$\mathbf{u} + \mathbf{v} + \mathbf{z} + \mathbf{w} = \mathbf{0} \quad \text{or} \quad \mathbf{u} + \mathbf{v} + \mathbf{z} = -\mathbf{w}.$$

This yields the following system of linear equations.

$$\begin{cases} \frac{2}{3}\|\mathbf{u}\| \qquad\qquad\quad - \frac{2}{3}\|\mathbf{z}\| = 0 \\ -\frac{2}{3}\|\mathbf{u}\| + \frac{2}{\sqrt{5}}\|\mathbf{v}\| - \frac{2}{3}\|\mathbf{z}\| = 0 \\ \frac{1}{3}\|\mathbf{u}\| + \frac{1}{\sqrt{5}}\|\mathbf{v}\| + \frac{1}{3}\|\mathbf{z}\| = 480 \end{cases}$$

Using the techniques demonstrated in Chapter 7, you can find the solution of the system to be

$$\|\mathbf{u}\| = 360.0$$

$$\|\mathbf{v}\| \approx 536.7$$

$$\|\mathbf{z}\| = 360.0.$$

So, the rope attached at point $P$ has 360 pounds of tension, the rope attached at point $Q$ has about 536.7 pounds of tension, and the rope attached at point $R$ has 360 pounds of tension.

CHECKPOINT Now try Exercise 71.

## 10.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

### *Vocabulary Check*

**Fill in the blanks.**

**1.** The _______ vector is denoted by $\mathbf{0} = \langle 0, 0, 0 \rangle$.

**2.** The standard unit vector notation for a vector $\mathbf{v}$ in space is given by _______ .

**3.** The _______ of a vector $\mathbf{v}$ is produced by subtracting the coordinates of the initial point from the corresponding coordinates of the terminal point.

**4.** If the dot product of two nonzero vectors is zero, the angle between the vectors is 90° and the vectors are called _______ .

**5.** Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are _______ if there is some scalar $c$ such that $\mathbf{u} = c\mathbf{v}$.

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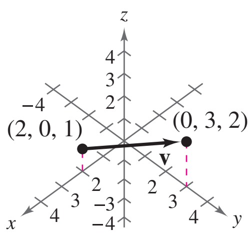
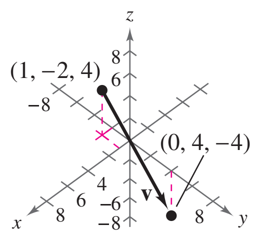
**In Exercises 1–4, (a) find the component form of the vector v and (b) sketch the vector with its initial point at the origin.**

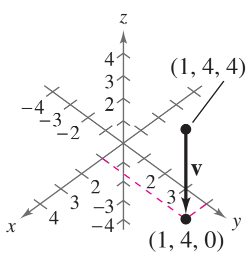
**1.**



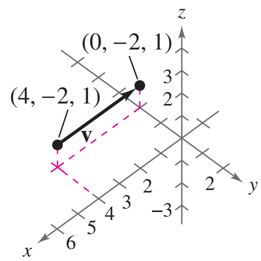

**2.**




**3.**




**4.**




**In Exercises 5–8, (a) write the component form of the vector v, (b) find the magnitude of v, and (c) find a unit vector in the direction of v.**

| | *Initial point* | *Terminal point* |
|---|---|---|
| **5.** | $(-6, 4, -2)$ | $(1, -1, 3)$ |
| **6.** | $(-7, 3, 5)$ | $(0, 0, 2)$ |
| **7.** | $(-1, 2, -4)$ | $(1, 4, -4)$ |
| **8.** | $(0, -1, 0)$ | $(0, 2, 1)$ |

**In Exercises 9–12, sketch each scalar multiple of v.**

**9.** $\mathbf{v} = \langle 1, 1, 3 \rangle$
(a) $2\mathbf{v}$ (b) $-\mathbf{v}$ (c) $\frac{3}{2}\mathbf{v}$ (d) $0\mathbf{v}$

**10.** $\mathbf{v} = \langle -1, 2, 2 \rangle$
(a) $-\mathbf{v}$ (b) $2\mathbf{v}$ (c) $\frac{1}{2}\mathbf{v}$ (d) $\frac{5}{2}\mathbf{v}$

**11.** $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
(a) $2\mathbf{v}$ (b) $-\mathbf{v}$ (c) $\frac{5}{2}\mathbf{v}$ (d) $0\mathbf{v}$

**12.** $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$
(a) $4\mathbf{v}$ (b) $-2\mathbf{v}$ (c) $\frac{1}{2}\mathbf{v}$ (d) $0\mathbf{v}$

**In Exercises 13–20, find the vector z, given $\mathbf{u} = \langle -1, 3, 2 \rangle$, $\mathbf{v} = \langle 1, -2, -2 \rangle$, and $\mathbf{w} = \langle 5, 0, -5 \rangle$. Use a graphing utility to verify your answer.**

**13.** $\mathbf{z} = \mathbf{u} - 2\mathbf{v}$

**14.** $\mathbf{z} = 7\mathbf{u} + \mathbf{v} - \frac{1}{5}\mathbf{w}$

**15.** $2\mathbf{z} - 4\mathbf{u} = \mathbf{w}$

**16.** $\mathbf{u} + \mathbf{v} + \mathbf{z} = \mathbf{0}$

**17.** $\mathbf{z} = 2\mathbf{u} - 3\mathbf{v} + \frac{1}{2}\mathbf{w}$

**18.** $\mathbf{z} = 3\mathbf{w} - 2\mathbf{v} + \mathbf{u}$

**19.** $4\mathbf{z} = 4w - \mathbf{u} + \mathbf{v}$

**20.** $\mathbf{u} + 2\mathbf{v} + \mathbf{z} = \mathbf{w}$

**In Exercises 21–30, find the magnitude of v.**

**21.** $\mathbf{v} = \langle 7, 8, 7 \rangle$

**22.** $\mathbf{v} = \langle -2, 0, -5 \rangle$

**23.** $\mathbf{v} = \langle 1, -2, 4 \rangle$

**24.** $\mathbf{v} = \langle -1, 0, 3 \rangle$

**25.** $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

**26.** $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$

**27.** $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$

**28.** $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$

**29.** Initial point: $(1, -3, 4)$; terminal point: $(1, 0, -1)$

**30.** Initial point: $(0, -1, 0)$; terminal point: $(1, 2, -2)$

**In Exercises 31–34, find a unit vector (a) in the direction of u and (b) in the direction opposite of u.**

**31.** $\mathbf{u} = 5\mathbf{i} - 12\mathbf{k}$

**32.** $\mathbf{u} = 3\mathbf{i} - 4\mathbf{k}$

**33.** $\mathbf{u} = 8\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

**34.** $\mathbf{u} = -3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$

**In Exercises 35–38, use a graphing utility to determine the specified quantity where $\mathbf{u} = \langle -1, 3, 4 \rangle$ and $\mathbf{v} = \langle 5, 4.5, -6 \rangle$.**

**35.** $6\mathbf{u} - 4\mathbf{v}$

**36.** $2\mathbf{u} + \frac{5}{2}\mathbf{v}$

**37.** $\|\mathbf{u} + \mathbf{v}\|$

**38.** $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

**In Exercises 39–42, find the dot product of u and v.**

**39.** $\mathbf{u} = \langle 4, 4, -1 \rangle$
$\mathbf{v} = \langle 2, -5, -8 \rangle$

**40.** $\mathbf{u} = \langle 3, -1, 6 \rangle$
$\mathbf{v} = \langle 4, -10, 1 \rangle$

**41.** $\mathbf{u} = 2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$
$\mathbf{v} = 9\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

**42.** $\mathbf{u} = 3\mathbf{j} - 6\mathbf{k}$
$\mathbf{v} = 6\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$

**In Exercises 43–46, find the angle $\theta$ between the vectors.**

**43.** $\mathbf{u} = \langle 0, 2, 2 \rangle$
$\mathbf{v} = \langle 3, 0, -4 \rangle$

**44.** $\mathbf{u} = \langle -1, 3, 0 \rangle$
$\mathbf{v} = \langle 1, 2, -1 \rangle$

**45.** $\mathbf{u} = 10\mathbf{i} + 40\mathbf{j}$
$\mathbf{v} = -3\mathbf{j} + 8\mathbf{k}$

**46.** $\mathbf{u} = 8\mathbf{j} - 20\mathbf{k}$
$\mathbf{v} = 10\mathbf{i} - 5\mathbf{k}$

**In Exercises 47–54, determine whether u and v are orthogonal, parallel, or neither.**

**47.** $\mathbf{u} = \langle -12, 6, 15 \rangle$
$\mathbf{v} = \langle 8, -4, -10 \rangle$

**48.** $\mathbf{u} = \langle -1, 3, -1 \rangle$
$\mathbf{v} = \langle 2, -1, 5 \rangle$

**49.** $\mathbf{u} = \frac{3}{4}\mathbf{i} - \frac{1}{2}\mathbf{j} + 2\mathbf{k}$
$\mathbf{v} = 4\mathbf{i} + 10\mathbf{j} + \mathbf{k}$

**50.** $\mathbf{u} = -\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$
$\mathbf{v} = 8\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$

**51.** $\mathbf{u} = \mathbf{j} + 6\mathbf{k}$
$\mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

**52.** $\mathbf{u} = 4\mathbf{i} - \mathbf{k}$
$\mathbf{v} = \mathbf{i}$

**53.** $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

**54.** $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
$\mathbf{v} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$

**In Exercises 55–58, use vectors to determine whether the points are collinear.**

**55.** $(5, 4, 1), (7, 3, -1), (4, 5, 3)$

**56.** $(-2, 7, 4), (-4, 8, 1), (0, 6, 7)$

**57.** $(1, 3, 2), (-1, 2, 5), (3, 4, -1)$

**58.** $(0, 4, 4), (-1, 5, 6), (-2, 6, 7)$

**In Exercises 59–62, the vertices of a triangle are given. Determine whether the traingle is an acute triangle, an obtuse triangle, or a right triangle. Explain your reasoning.**

**59.** $(1, 2, 0), (0, 0, 0), (-2, 1, 0)$

**60.** $(-3, 0, 0), (0, 0, 0), (1, 2, 3)$

**61.** $(2, -3, 4), (0, 1, 2), (-1, 2, 0)$

**62.** $(2, -7, 3), (-1, 5, 8), (4, 6, -1)$

**In Exercises 63–66, the vector v and its initial point are given. Find the terminal point.**

**63.** $\mathbf{v} = \langle 2, -4, 7 \rangle$
Initial point: $(1, 5, 0)$

**64.** $\mathbf{v} = \langle 4, -1, -1 \rangle$
Initial point: $(6, -4, 3)$

**65.** $\mathbf{v} = \left\langle 4, \frac{3}{2}, -\frac{1}{4} \right\rangle$
Initial point: $\left(2, 1, -\frac{3}{2}\right)$

**66.** $\mathbf{v} = \left\langle \frac{5}{2}, -\frac{1}{2}, 4 \right\rangle$
Initial point: $\left(3, 2, -\frac{1}{2}\right)$

**67.** Determine the values of $c$ such that $\|c\mathbf{u}\| = 3$, where $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

**68.** Determine the values of $c$ such that $\|c\mathbf{u}\| = 12$, where $\mathbf{u} = -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

**In Exercises 69 and 70, write the component form of v.**

**69.** $\mathbf{v}$ lies in the $yz$-plane, has magnitude 4, and makes an angle of 45° with the positive $y$-axis.

**70.** $\mathbf{v}$ lies in the $xz$-plane, has magnitude 10, and makes an angle of 60° with the positive $z$-axis.

**71.** ***Tension*** The lights in an auditorium are 30-pound disks of radius 24 inches. Each disk is supported by three equally spaced 60-inch wires attached to the ceiling (see figure). Find the tension in each wire.

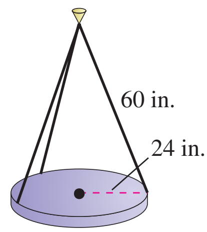



Figure for 71

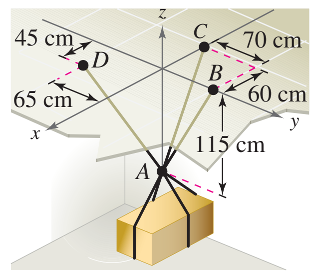



Figure for 72

**72.** ***Tension*** The weight of a crate is 500 newtons. Find the tension in each of the supporting cables shown in the figure.

## Synthesis

***True or False?*** **In Exercises 73 and 74, determine whether the statement is true or false. Justify your answer.**

**73.** If the dot product of two nonzero vectors is zero, then the angle between the vectors is a right angle.

**74.** If $AB$ and $AC$ are parallel vectors, then points $A$, $B$, and $C$ are collinear.

**75.** ***Exploration*** Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$, and $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$.

(a) Sketch $\mathbf{u}$ and $\mathbf{v}$.

(b) If $\mathbf{w} = \mathbf{0}$, show that $a$ and $b$ must both be zero.

(c) Find $a$ and $b$ such that $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(d) Show that no choice of $a$ and $b$ yields $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

**76.** ***Think About It*** The initial and terminal points of $\mathbf{v}$ are $(x_1, y_1, z_1)$ and $(x, y, z)$, respectively. Describe the set of all points $(x, y, z)$ such that $\|\mathbf{v}\| = 4$.

**77.** What is known about the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ if $\mathbf{u} \cdot \mathbf{v} < 0$? Explain.

**78.** ***Writing*** Consider the two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$. Describe the geometric figure generated by the terminal points of the vectors $t\mathbf{v}$, $\mathbf{u} + t\mathbf{v}$, and $s\mathbf{u} + t\mathbf{v}$, where $s$ and $t$ represent real numbers.

# 10.3 The Cross Product of Two Vectors

## The Cross Product

Many applications in physics, engineering, and geometry involve finding a vector in space that is orthogonal to two given vectors. In this section, you will study a product that will yield such a vector. It is called the **cross product,** and it is most conveniently defined and calculated using the standard unit vector form.

### Definition of Cross Product of Two Vectors in Space

Let

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \qquad \text{and} \qquad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

be vectors in space. The **cross product** of **u** and **v** is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

It is important to note that this definition applies only to three-dimensional vectors. The cross product is not defined for two-dimensional vectors.

A convenient way to calculate $\mathbf{u} \times \mathbf{v}$ is to use the following *determinant form* with cofactor expansion. (This $3 \times 3$ determinant form is used simply to help remember the formula for the cross product—it is technically not a determinant because the entries of the corresponding matrix are not all real numbers.)

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Put **u** in Row 2.
Put **v** in Row 3.

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Note the minus sign in front of the **j**-component. Recall from Section 7.7 that each of the three $2 \times 2$ determinants can be evaluated by using the following pattern.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

### Exploration

Find each cross product. What can you conclude?

**a.** $\mathbf{i} \times \mathbf{j}$ **b.** $\mathbf{i} \times \mathbf{k}$ **c.** $\mathbf{j} \times \mathbf{k}$

### What you should learn

- Find cross products of vectors in space.
- Use geometric properties of cross products of vectors in space.
- Use triple scalar products to find volumes of parallelepipeds.

### Why you should learn it

The cross product of two vectors in space has many applications in physics and engineering. For instance, in Exercise 57 on page 763, the cross product is used to find the torque on the crank of a bicycle's brake.

Alamy

## Example 1 Finding Cross Products

Given $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, find each cross product.

**a.** $\mathbf{u} \times \mathbf{v}$ **b.** $\mathbf{v} \times \mathbf{u}$ **c.** $\mathbf{v} \times \mathbf{v}$

### Solution

**a.** $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}\mathbf{k}$$

$$= (4 - 1)\mathbf{i} - (2 - 3)\mathbf{j} + (1 - 6)\mathbf{k}$$

$$= 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$$

**b.** $$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{k}$$

$$= (1 - 4)\mathbf{i} - (3 - 2)\mathbf{j} + (6 - 1)\mathbf{k}$$

$$= -3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

Note that this result is the negative of that in part (a).

**c.** $$\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 3 & 1 & 2 \end{vmatrix} = \mathbf{0}$$

CHECKPOINT Now try Exercise 1.

**TECHNOLOGY TIP**

Some graphing utilities have the capability to perform vector operations, such as the cross product. Consult the user's guide for your graphing utility for specific instructions.

**Exploration**

Calculate $\mathbf{u} \times \mathbf{v}$ and $-(\mathbf{v} \times \mathbf{u})$ for several values of $\mathbf{u}$ and $\mathbf{v}$. What do your results imply? Interpret your results geometrically.

The results obtained in Example 1 suggest some interesting algebraic properties of the cross product. For instance,

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \quad \text{and} \quad \mathbf{v} \times \mathbf{v} = \mathbf{0}.$$

These properties, and several others, are summarized in the following list.

**Algebraic Properties of the Cross Product** (See the proof on page 777.)

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space and let $c$ be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

## Geometric Properties of the Cross Product

The first property listed on the preceding page indicates that the cross product is *not commutative*. In particular, this property indicates that the vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ have equal lengths but opposite directions. The following list gives some other *geometric* properties of the cross product of two vectors.

> **Geometric Properties of the Cross Product** (See the proof on page 778.)
>
> Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let $\theta$ be the angle between $\mathbf{u}$ and $\mathbf{v}$.
>
> 1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
> 2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin \theta$
> 3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.
> 4. $\|\mathbf{u} \times \mathbf{v}\| =$ area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

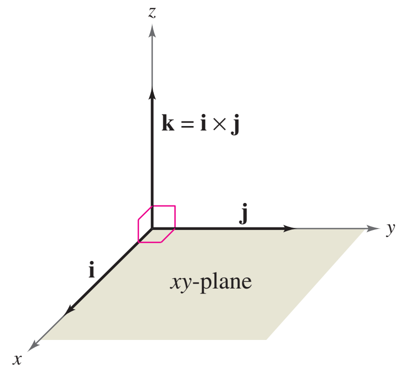


**Figure 10.21**

Both $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ are perpendicular to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. One way to remember the orientations of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is to compare them with the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k} = \mathbf{i} \times \mathbf{j}$, as shown in Figure 10.21. The three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a *right-handed system*.

### Example 2 Using the Cross Product

Find a unit vector that is orthogonal to both

$$\mathbf{u} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k} \qquad \text{and} \qquad \mathbf{v} = -3\mathbf{i} + 6\mathbf{j}.$$

#### Solution

The cross product $\mathbf{u} \times \mathbf{v}$, as shown in Figure 10.22, is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 1 \\ -3 & 6 & 0 \end{vmatrix}$$

$$= -6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$$

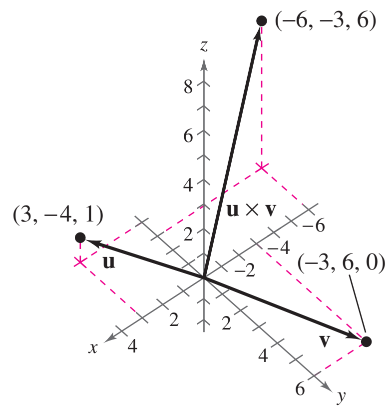


**Figure 10.22**

Because

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-6)^2 + (-3)^2 + 6^2}$$

$$= \sqrt{81}$$

$$= 9$$

a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

CHECKPOINT Now try Exercise 29.

In Example 2, note that you could have used the cross product $\mathbf{v} \times \mathbf{u}$ to form a unit vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. With that choice, you would have obtained the *negative* of the unit vector found in the example.

The fourth geometric property of the cross product states that $\|\mathbf{u} \times \mathbf{v}\|$ is the area of the parallelogram that has $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides. A simple example of this is given by the unit square with adjacent sides of $\mathbf{i}$ and $\mathbf{j}$. Because

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

and $\|\mathbf{k}\| = 1$, it follows that the square has an area of 1. This geometric property of the cross product is illustrated further in the next example.

## Example 3 Geometric Application of the Cross Product

Show that the quadrilateral with vertices at the following points is a parallelogram. Then find the area of the parallelogram. Is the parallelogram a rectangle?

$$A(5, 2, 0), \qquad B(2, 6, 1), \qquad C(2, 4, 7), \qquad D(5, 0, 6)$$

### Solution

From Figure 10.23 you can see that the sides of the quadrilateral correspond to the following four vectors.

$$\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

$$\overrightarrow{CD} = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k} = -\overrightarrow{AB}$$

$$\overrightarrow{AD} = 0\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

$$\overrightarrow{CB} = 0\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} = -\overrightarrow{AD}$$

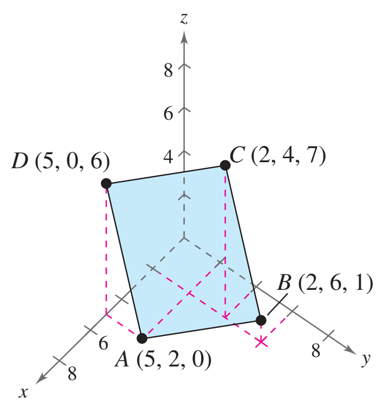



**Figure 10.23**

Because $\overrightarrow{CD} = -\overrightarrow{AB}$ and $\overrightarrow{CB} = -\overrightarrow{AD}$, you can conclude that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and $\overrightarrow{AD}$ is parallel to $\overrightarrow{CB}$. It follows that the quadrilateral is a parallelogram with $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as adjacent sides. Moreover, because

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} = 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}$$

the area of the parallelogram is

$$\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{26^2 + 18^2 + 6^2} = \sqrt{1036} \approx 32.19.$$

You can tell whether the parallelogram is a rectangle by finding the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

$$\sin\theta = \frac{\|\overrightarrow{AB} \times \overrightarrow{AD}\|}{\|\overrightarrow{AB}\|\,\|\overrightarrow{AD}\|}$$

$$= \frac{\sqrt{1036}}{\sqrt{26}\sqrt{40}} \approx 0.998$$

$$\theta = \arcsin 0.998$$

$$\theta \approx 86.4^\circ$$

Because $\theta \neq 90^\circ$, the parallelogram is not a rectangle.

CHECKPOINT Now try Exercise 41.

### Exploration

If you connect the terminal points of two vectors $\mathbf{u}$ and $\mathbf{v}$ that have the same initial points, a triangle is formed. Is it possible to use the cross product $\mathbf{u} \times \mathbf{v}$ to determine the area of the triangle? Explain. Verify your conclusion using two vectors from Example 3.

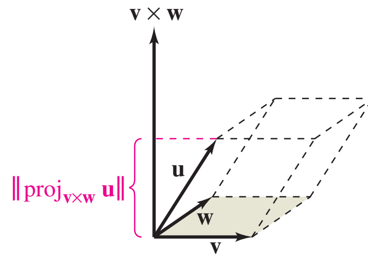
## The Triple Scalar Product

For the vectors **u**, **v**, and **w** in space, the dot product of **u** and $\mathbf{v} \times \mathbf{w}$ is called the **triple scalar product** of **u**, **v**, and **w**.

### The Triple Scalar Product

For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, the **triple scalar product** is given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$




*Area of base* $= \|\mathbf{v} \times \mathbf{w}\|$
*Volume of parallelepiped* $= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$
**Figure 10.24**

If the vectors **u**, **v**, and **w** do not lie in the same plane, the triple scalar product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ can be used to determine the volume of the parallelepiped (a polyhedron, all of whose faces are parallelograms) with **u**, **v**, and **w** as adjacent edges, as shown in Figure 10.24.

### Geometric Property of Triple Scalar Product

The volume $V$ of a parallelepiped with vectors **u**, **v**, and **w** as adjacent edges is given by

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

### Example 4 Volume by the Triple Scalar Product

Find the volume of the parallelepiped having

$$\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}, \quad \mathbf{v} = 2\mathbf{j} - 2\mathbf{k}, \quad \text{and} \quad \mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$$

as adjacent edges, as shown in Figure 10.25.

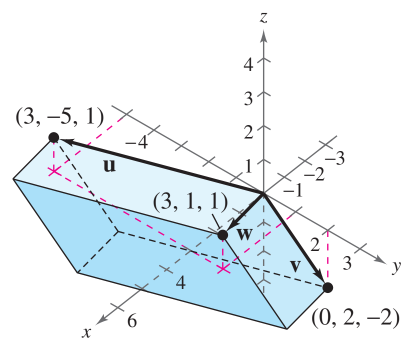



**Figure 10.25**

#### Solution

The value of the triple scalar product is

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} \\
&= 3\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\
&= 3(4) + 5(6) + 1(-6) \\
&= 36.
\end{aligned}$$

So, the volume of the parallelepiped is

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |36| = 36.$$

CHECKPOINT Now try Exercise 53.

## 10.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

### Vocabulary Check

**Fill in the blanks.**

1. To find a vector in space that is orthogonal to two given vectors, find the _______ of the two vectors.
2. $\mathbf{u} \times \mathbf{u} =$ _______
3. $\|\mathbf{u} \times \mathbf{v}\| =$ _______
4. The dot product of $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$ is called the _______ of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

---

**In Exercises 1–4, find the cross product of the unit vectors and sketch the result.**

1. $\mathbf{j} \times \mathbf{i}$
2. $\mathbf{k} \times \mathbf{j}$
3. $\mathbf{i} \times \mathbf{k}$
4. $\mathbf{k} \times \mathbf{i}$

**In Exercises 5–20, find u × v and show that it is orthogonal to both u and v.**

5. $\mathbf{u} = \langle 1, -1, 0 \rangle$, $\mathbf{v} = \langle 0, 1, -1 \rangle$
6. $\mathbf{u} = \langle -1, 1, 0 \rangle$, $\mathbf{v} = \langle 1, 0, -1 \rangle$
7. $\mathbf{u} = \langle 3, -2, 5 \rangle$, $\mathbf{v} = \langle 0, -1, 1 \rangle$
8. $\mathbf{u} = \langle 2, -3, 1 \rangle$, $\mathbf{v} = \langle 1, -2, 1 \rangle$
9. $\mathbf{u} = \langle -10, 0, 6 \rangle$, $\mathbf{v} = \langle 7, 0, 0 \rangle$
10. $\mathbf{u} = \langle -5, 5, 11 \rangle$, $\mathbf{v} = \langle 2, 2, 3 \rangle$
11. $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$
12. $\mathbf{u} = \mathbf{i} + \frac{3}{2}\mathbf{j} - \frac{5}{2}\mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{4}\mathbf{k}$
13. $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$
14. $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
15. $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \mathbf{k}$, $\mathbf{v} = -\frac{3}{4}\mathbf{i} + \mathbf{j} + \frac{1}{4}\mathbf{k}$
16. $\mathbf{u} = \frac{2}{5}\mathbf{i} - \frac{1}{4}\mathbf{j} + \frac{1}{2}\mathbf{k}$, $\mathbf{v} = -\frac{3}{5}\mathbf{i} + \mathbf{j} + \frac{1}{5}\mathbf{k}$
17. $\mathbf{u} = 6\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
18. $\mathbf{u} = \frac{2}{3}\mathbf{i}$, $\mathbf{v} = \frac{1}{3}\mathbf{j} - 3\mathbf{k}$
19. $\mathbf{u} = -\mathbf{i} + \mathbf{k}$, $\mathbf{v} = \mathbf{j} - 2\mathbf{k}$
20. $\mathbf{u} = \mathbf{i} - 2\mathbf{k}$, $\mathbf{v} = -\mathbf{j} + \mathbf{k}$

**In Exercises 21–26, use a graphing utility to find u × v.**

21. $\mathbf{u} = \langle 2, 4, 3 \rangle$, $\mathbf{v} = \langle 0, -2, 1 \rangle$
22. $\mathbf{u} = \langle 4, -2, 6 \rangle$, $\mathbf{v} = \langle -1, 5, 7 \rangle$
23. $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = -4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
24. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} - 4\mathbf{k}$
25. $\mathbf{u} = 6\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{2}{10}\mathbf{k}$
26. $\mathbf{u} = 8\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{i} + \frac{3}{4}\mathbf{j} - \frac{1}{4}\mathbf{k}$

**In Exercises 27–34, find a unit vector orthogonal to u and v.**

27. $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 2, -3, 0 \rangle$
28. $\mathbf{u} = \langle 2, -1, 3 \rangle$, $\mathbf{v} = \langle 1, 0, -2 \rangle$
29. $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$
30. $\mathbf{u} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} - 3\mathbf{k}$
31. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{10}\mathbf{k}$
32. $\mathbf{u} = 7\mathbf{i} - 14\mathbf{j} + 5\mathbf{k}$, $\mathbf{v} = 14\mathbf{i} + 28\mathbf{j} - 15\mathbf{k}$
33. $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
34. $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

**In Exercises 35–40, find the area of the parallelogram that has the vectors as adjacent sides.**

35. $\mathbf{u} = \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{k}$
36. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{k}$
37. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$
38. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$
39. $\mathbf{u} = \langle 2, 2, -3 \rangle$, $\mathbf{v} = \langle 0, 2, 3 \rangle$
40. $\mathbf{u} = \langle 4, -3, 2 \rangle$, $\mathbf{v} = \langle 5, 0, 1 \rangle$

**In Exercises 41 and 42, (a) verify that the points are the vertices of a parallelogram, (b) find its area, and (c) determine whether the parallelogram is a rectangle.**

41. $A(2, -1, 4)$, $B(3, 1, 2)$, $C(0, 5, 6)$, $D(-1, 3, 8)$
42. $A(1, 1, 1)$, $B(2, 3, 4)$, $C(6, 5, 2)$, $D(7, 7, 5)$

**In Exercises 43–46, find the area of the triangle with the given vertices. (The area $A$ of the triangle having u and v as adjacent sides is given by $A = \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\|$.)**

43. $(0, 0, 0)$, $(1, 2, 3)$, $(-3, 0, 0)$
44. $(1, -4, 3)$, $(2, 0, 2)$, $(-2, 2, 0)$
45. $(2, 3, -5)$, $(-2, -2, 0)$, $(3, 0, 6)$
46. $(2, 4, 0)$, $(-2, -4, 0)$, $(0, 0, 4)$

**In Exercises 47–50, find the triple scalar product.**

**47.** $\mathbf{u} = \langle 2, 3, 3\rangle$, $\mathbf{v} = \langle 4, 4, 0\rangle$, $\mathbf{w} = \langle 0, 0, 4\rangle$

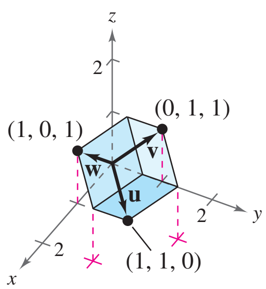
**48.** $\mathbf{u} = \langle 2, 0, 1\rangle$, $\mathbf{v} = \langle 0, 3, 0\rangle$, $\mathbf{w} = \langle 0, 0, 1\rangle$

**49.** $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$, $\mathbf{w} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

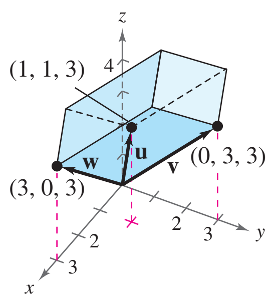
**50.** $\mathbf{u} = \mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 4\mathbf{k}$, $\mathbf{w} = -3\mathbf{j} + 6\mathbf{k}$

**In Exercises 51–54, use the triple scalar product to find the volume of the parallelepiped having adjacent edges u, v, and w.**

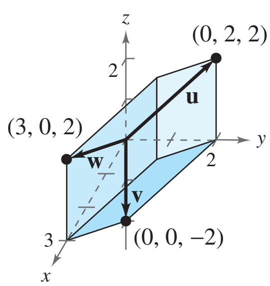
**51.** $\mathbf{u} = \mathbf{i} + \mathbf{j}$
$\mathbf{v} = \mathbf{j} + \mathbf{k}$
$\mathbf{w} = \mathbf{i} + \mathbf{k}$



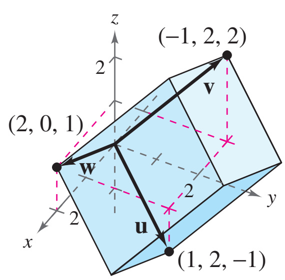

**52.** $\mathbf{u} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$
$\mathbf{v} = 3\mathbf{j} + 3\mathbf{k}$
$\mathbf{w} = 3\mathbf{i} + 3\mathbf{k}$




**53.** $\mathbf{u} = \langle 0, 2, 2\rangle$
$\mathbf{v} = \langle 0, 0, -2\rangle$
$\mathbf{w} = \langle 3, 0, 2\rangle$




**54.** $\mathbf{u} = \langle 1, 2, -1\rangle$
$\mathbf{v} = \langle -1, 2, 2\rangle$
$\mathbf{w} = \langle 2, 0, 1\rangle$




**In Exercises 55 and 56, find the volume of the parallelepiped with the given vertices.**

**55.** $A(0, 0, 0)$, $B(4, 0, 0)$, $C(4, -2, 3)$, $D(0, -2, 3)$,
$E(4, 5, 3)$, $F(0, 5, 3)$, $G(0, 3, 6)$, $H(4, 3, 6)$

**56.** $A(0, 0, 0)$, $B(1, 1, 0)$, $C(1, 0, 2)$, $D(0, 1, 1)$,
$E(2, 1, 2)$, $F(1, 1, 3)$, $G(1, 2, 1)$, $H(2, 2, 3)$

**57.** ***Torque*** The brakes on a bicycle are applied by using a downward force of $p$ pounds on the pedal when the six-inch crank makes a 40° angle with the horizontal (see figure). Vectors representing the position of the crank and the force are $\mathbf{V} = \frac{1}{2}(\cos 40°\,\mathbf{j} + \sin 40°\,\mathbf{k})$ and $\mathbf{F} = -p\mathbf{k}$, respectively.

(a) The magnitude of the torque on the crank is given by $\|\mathbf{V} \times \mathbf{F}\|$. Using the given information, write the torque $T$ on the crank as a function of $p$.

(b) Use the function from part (a) to complete the table.

| $p$ | 15 | 20 | 25 | 30 | 35 | 40 | 45 |
|---|---|---|---|---|---|---|---|
| $T$ | | | | | | | |

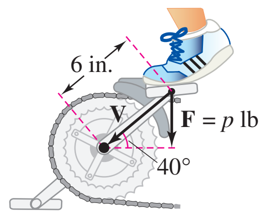



Figure for 57

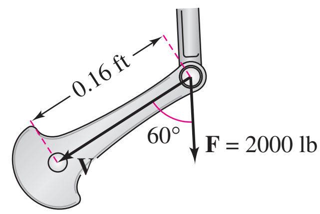



Figure for 58

**58.** ***Torque*** Both the magnitude and direction of the force on a crankshaft change as the crankshaft rotates. Use the technique given in Exercise 57 to find the magnitude of the torque on the crankshaft using the position and data shown in the figure.

## *Synthesis*

***True or False?*** **In Exercises 59 and 60, determine whether the statement is true or false. Justify your answer.**

**59.** The cross product is not defined for vectors in the plane.

**60.** If $\mathbf{u}$ and $\mathbf{v}$ are vectors in space that are nonzero and not parallel, then $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

**61.** ***Proof*** Prove that $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|$ if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

**62.** ***Proof*** Prove that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

**63.** ***Proof*** Prove that the triple scalar product of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

**64.** ***Proof*** Consider the vectors $\mathbf{u} = \langle \cos\alpha, \sin\alpha, 0\rangle$ and $\mathbf{v} = \langle \cos\beta, \sin\beta, 0\rangle$, where $\alpha > \beta$. Find the cross product of the vectors and use the result to prove the identity $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

## *Skills Review*

**In Exercises 65–68, evaluate the expression without using a calculator.**

**65.** $\cos 480°$

**66.** $\tan 300°$

**67.** $\sin \dfrac{19\pi}{6}$

**68.** $\cos \dfrac{17\pi}{6}$

# 10.4 Lines and Planes in Space

## Lines in Space

In the plane, *slope* is used to determine an equation of a line. In space, it is more convenient to use *vectors* to determine the equation of a line.

In Figure 10.26, consider the line $L$ through the point $P(x_1, y_1, z_1)$ and parallel to the vector

$\mathbf{v} = \langle a, b, c \rangle$. Direction vector for $L$

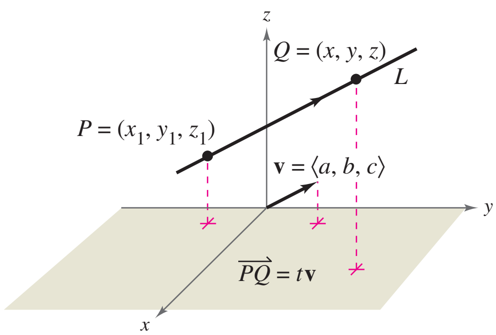



**Figure 10.26**

The vector $\mathbf{v}$ is the **direction vector** for the line $L$, and $a$, $b$, and $c$ are the **direction numbers.** One way of describing the line $L$ is to say that it consists of all points $Q(x, y, z)$ for which the vector $\overrightarrow{PQ}$ is parallel to $\mathbf{v}$. This means that $\overrightarrow{PQ}$ is a scalar multiple of $\mathbf{v}$, and you can write $\overrightarrow{PQ} = t\mathbf{v}$, where $t$ is a scalar.

$$\begin{aligned}\overrightarrow{PQ} &= \langle x - x_1, y - y_1, z - z_1 \rangle \\ &= \langle at, bt, ct \rangle \\ &= t\mathbf{v}\end{aligned}$$

By equating corresponding components, you can obtain the **parametric equations of a line in space.**

### Parametric Equations of a Line in Space

A line $L$ parallel to the nonzero vector $\mathbf{v} = \langle a, b, c \rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the parametric equations

$$x = x_1 + at, \qquad y = y_1 + bt, \qquad \text{and} \qquad z = z_1 + ct.$$

If the direction numbers $a$, $b$, and $c$ are all nonzero, you can eliminate the parameter $t$ to obtain the **symmetric equations** of a line.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$ Symmetric equations

### What you should learn

- Find parametric and symmetric equations of lines in space.
- Find equations of planes in space.
- Sketch planes in space.
- Find distances between points and planes in space.

### Why you should learn it

Normal vectors to a plane are important in modeling and solving real-life problems. For instance, in Exercise 60 on page 772, normal vectors are used to find the angle between two adjacent sides of a chute on a grain elevator of a combine.

Paul A. Souders/Corbis

## Example 1 Finding Parametric and Symmetric Equations

Find parametric and symmetric equations of the line $L$ that passes through the point $(1, -2, 4)$ and is parallel to $\mathbf{v} = \langle 2, 4, -4 \rangle$.

### Solution

To find a set of parametric equations of the line, use the coordinates $x_1 = 1$, $y_1 = -2$, and $z_1 = 4$ and direction numbers $a = 2$, $b = 4$, and $c = -4$ (see Figure 10.27).

$$x = 1 + 2t, \qquad y = -2 + 4t, \qquad z = 4 - 4t \qquad \text{Parametric equations}$$

Because $a$, $b$, and $c$ are all nonzero, a set of symmetric equations is

$$\frac{x-1}{2} = \frac{y+2}{4} = \frac{z-4}{-4}. \qquad \text{Symmetric equations}$$

CHECKPOINT Now try Exercise 1.

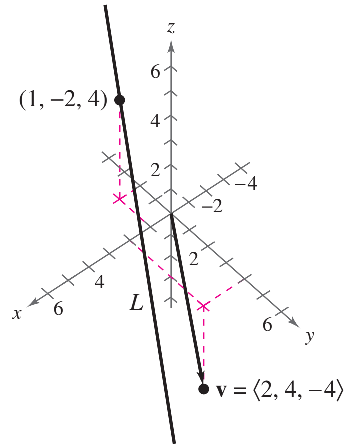



**Figure 10.27**

Neither the parametric equations nor the symmetric equations of a given line are unique. For instance, in Example 1, by letting $t = 1$ in the parametric equations you would obtain the point $(3, 2, 0)$. Using this point with the direction numbers $a = 2$, $b = 4$, and $c = -4$ produces the parametric equations

$$x = 3 + 2t, \qquad y = 2 + 4t, \quad \text{and} \quad z = -4t.$$

## Example 2 Parametric and Symmetric Equations of a Line Through Two Points

Find a set of parametric and symmetric equations of the line that passes through the points $(-2, 1, 0)$ and $(1, 3, 5)$.

### Solution

Begin by letting $P = (-2, 1, 0)$ and $Q = (1, 3, 5)$. Then a direction vector for the line passing through $P$ and $Q$ is

$$\begin{aligned}\mathbf{v} &= \overrightarrow{PQ}\\ &= \langle 1 - (-2), 3 - 1, 5 - 0 \rangle\\ &= \langle 3, 2, 5 \rangle\\ &= \langle a, b, c \rangle.\end{aligned}$$

Using the direction numbers $a = 3$, $b = 2$, and $c = 5$ with the point $P(-2, 1, 0)$, you can obtain the parametric equations

$$x = -2 + 3t, \qquad y = 1 + 2t, \quad \text{and} \quad z = 5t. \qquad \text{Parametric equations}$$

Because $a$, $b$, and $c$ are all nonzero, a set of symmetric equations is

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{z}{5}. \qquad \text{Symmetric equations}$$

CHECKPOINT Now try Exercise 7.

### STUDY TIP

To check the answer to Example 2, verify that the two original points lie on the line. To see this, substitute $t = 0$ and $t = 1$ into the parametric equations as follows.

$t = 0$:

$x = -2 + 3(0) = -2$

$y = 1 + 2(0) = 1$

$z = 5(0) = 0$

$t = 1$:

$x = -2 + 3(1) = 1$

$y = 1 + 2(1) = 3$

$z = 5(1) = 5$

## Planes in Space

You have seen how an equation of a line in space can be obtained from a point on the line and a vector *parallel* to it. You will now see that an equation of a plane in space can be obtained from a point in the plane and a vector *normal* (perpendicular) to the plane.

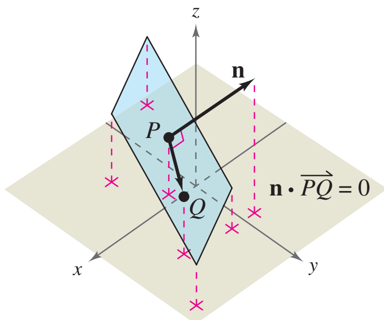



**Figure 10.28**

Consider the plane containing the point $P(x_1, y_1, z_1)$ having a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$, as shown in Figure 10.28. This plane consists of all points $Q(x, y, z)$ for which the vector $\overrightarrow{PQ}$ is orthogonal to $\mathbf{n}$. Using the dot product, you can write the following.

$$\mathbf{n} \cdot \overrightarrow{PQ} = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

The third equation of the plane is said to be in **standard form.**

### Standard Equation of a Plane in Space

The plane containing the point $(x_1, y_1, z_1)$ and having nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$ can be represented by the **standard form of the equation of a plane** $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

Regrouping terms yields the **general form of the equation of a plane** in space

$$ax + by + cz + d = 0. \qquad \text{General form of equation of plane}$$

Given the general form of the equation of a plane, it is easy to find a normal vector to the plane. Use the coefficients of $x$, $y$, and $z$ to write $\mathbf{n} = \langle a, b, c \rangle$.

### Exploration

Consider the following four planes.

$$2x + 3y - z = 2 \qquad 4x + 6y - 2z = 5$$

$$-2x - 3y + z = -2 \qquad -6x - 9y + 3z = 11$$

What are the normal vectors for each plane? What can you say about the relative positions of these planes in space?

## Example 3 Finding an Equation of a Plane in Three-Space

Find the general equation of the plane passing through the points $(2, 1, 1)$, $(0, 4, 1)$, and $(-2, 1, 4)$.

### Solution

To find the equation of the plane, you need a point in the plane and a vector that is normal to the plane. There are three choices for the point, but no normal vector is given. To obtain a normal vector, use the cross product of vectors **u** and **v** extending from the point $(2, 1, 1)$ to the points $(0, 4, 1)$ and $(-2, 1, 4)$, as shown in Figure 10.29. The component forms of **u** and **v** are

$$\mathbf{u} = \langle 0 - 2, 4 - 1, 1 - 1 \rangle = \langle -2, 3, 0 \rangle$$

$$\mathbf{v} = \langle -2 - 2, 1 - 1, 4 - 1 \rangle = \langle -4, 0, 3 \rangle$$

and it follows that

$$\begin{aligned}\mathbf{n} = \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix} \\ &= 9\mathbf{i} + 6\mathbf{j} + 12\mathbf{k} \\ &= \langle a, b, c \rangle\end{aligned}$$

is normal to the given plane. Using the direction numbers for **n** and the point $(x_1, y_1, z_1) = (2, 1, 1)$, you can determine an equation of the plane to be

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$9(x - 2) + 6(y - 1) + 12(z - 1) = 0 \qquad \text{Standard form}$$

$$9x + 6y + 12z - 36 = 0$$

$$3x + 2y + 4z - 12 = 0. \qquad \text{General form}$$

Check that each of the three points satisfies the equation $3x + 2y + 4z - 12 = 0$.

CHECKPOINT Now try Exercise 25.

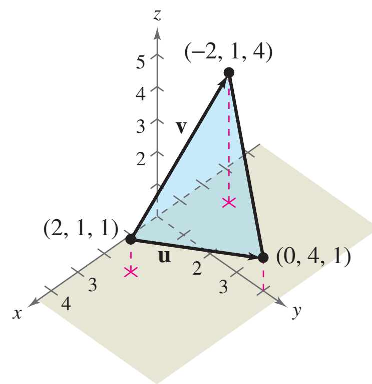



Figure 10.29

Two distinct planes in three-space either are parallel or intersect in a line. If they intersect, you can determine the angle $\theta$ $(0 \le \theta \le 90°)$ between them from the angle between their normal vectors, as shown in Figure 10.30. Specifically, if vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal to two intersecting planes, the angle $\theta$ between the normal vectors is equal to the **angle between the two planes** and is given by

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \, \|\mathbf{n}_2\|}. \qquad \text{Angle between two planes}$$

Consequently, two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are

1. *perpendicular* if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.
2. *parallel* if $\mathbf{n}_1$ is a scalar multiple of $\mathbf{n}_2$.

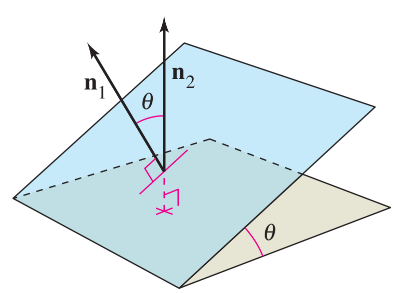



Figure 10.30

## Example 4 Finding the Line of Intersection of Two Planes

Find the angle between the two planes given by

$$x - 2y + z = 0 \qquad \text{Equation for plane 1}$$

$$2x + 3y - 2z = 0 \qquad \text{Equation for plane 2}$$

and find parametric equations of their line of intersection (see Figure 10.31).

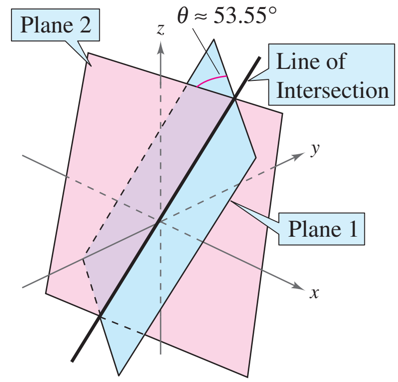



**Figure 10.31**

### Solution

The normal vectors for the planes are $\mathbf{n}_1 = \langle 1, -2, 1\rangle$ and $\mathbf{n}_2 = \langle 2, 3, -2\rangle$. Consequently, the angle between the two planes is determined as follows.

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{|-6|}{\sqrt{6}\sqrt{17}} = \frac{6}{\sqrt{102}} \approx 0.59409.$$

This implies that the angle between the two planes is $\theta \approx 53.55°$. You can find the line of intersection of the two planes by simultaneously solving the two linear equations representing the planes. One way to do this is to multiply the first equation by $-2$ and add the result to the second equation.

$$\begin{array}{r} x - 2y + z = 0 \\ \underline{2x + 3y - 2z = 0} \end{array} \quad \Rightarrow \quad \begin{array}{r} -2x + 4y - 2z = 0 \\ \underline{2x + 3y - 2z = 0} \\ 7y - 4z = 0 \end{array} \quad \Rightarrow \quad y = \frac{4z}{7}$$

Substituting $y = 4z/7$ back into one of the original equations, you can determine that $x = z/7$. Finally, by letting $t = z/7$, you obtain the parametric equations

$$x = t = x_1 + at, \qquad y = 4t = y_1 + bt, \qquad z = 7t = z_1 + ct.$$

Because $(x_1, y_1, z_1) = (0, 0, 0)$ lies in both planes, you can substitute for $x_1$, $y_1$, and $z_1$ in these parametric equations, which indicates that $a = 1$, $b = 4$, and $c = 7$ are direction numbers for the line of intersection.

CHECKPOINT Now try Exercise 45.

Note that the direction numbers in Example 4 can be obtained from the cross product of the two normal vectors as follows.

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 3 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}\mathbf{k}$$

$$= \mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$$

This means that the *line of intersection of the two planes is parallel to the cross product of their normal vectors*.

## Sketching Planes in Space

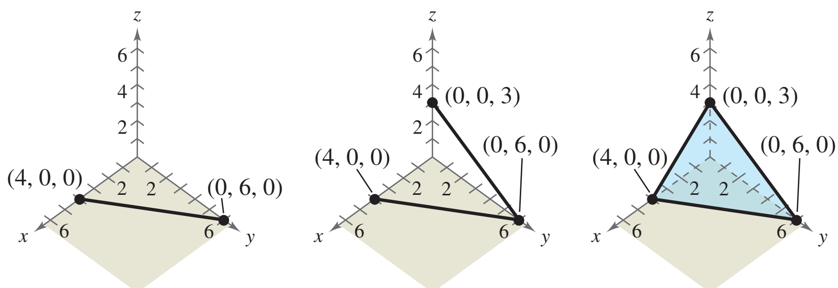
As discussed in Section 10.1, if a plane in space intersects one of the coordinate planes, the line of intersection is called the *trace* of the given plane in the coordinate plane. To sketch a plane in space, it is helpful to find its points of intersection with the coordinate axes and its traces in the coordinate planes. For example, consider the plane

$$3x + 2y + 4z = 12. \qquad \text{Equation of plane}$$

You can find the $xy$-trace by letting $z = 0$ and sketching the line

$$3x + 2y = 12 \qquad xy\text{-trace}$$

in the $xy$-plane. This line intersects the $x$-axis at $(4, 0, 0)$ and the $y$-axis at $(0, 6, 0)$. In Figure 10.32, this process is continued by finding the $yz$-trace and the $xz$-trace and then shading the triangular region lying in the first octant.




**(a)** ***xy*-trace** $(z = 0)$: $3x + 2y = 12$ **(b)** ***yz*-trace** $(x = 0)$: $2y + 4z = 12$ **(c)** ***xz*-trace** $(y = 0)$: $3x + 4z = 12$

**Figure 10.32**

If the equation of a plane has a missing variable, such as $2x + z = 1$, the plane must be *parallel to the axis* represented by the missing variable, as shown in Figure 10.33. If two variables are missing from the equation of a plane, then it is *parallel to the coordinate plane* represented by the missing variables, as shown in Figure 10.34.

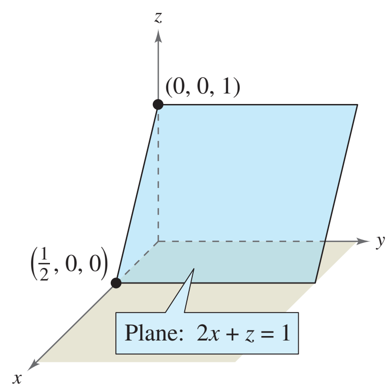
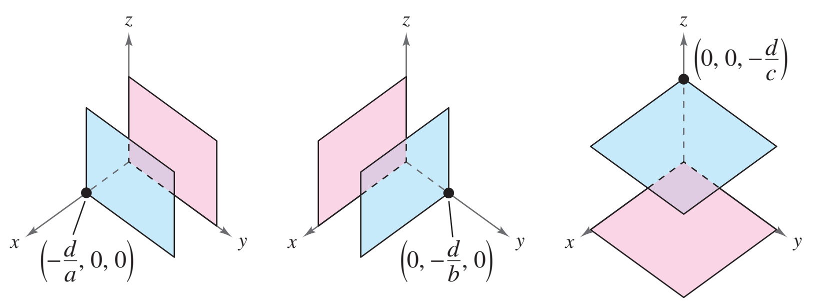



**Figure 10.33** ***Plane is parallel to y-axis.***

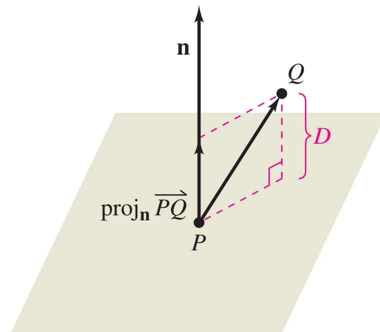



**(a)** ***Plane*** $ax + d = 0$ ***is parallel to yz-plane.*** **(b)** ***Plane*** $by + d = 0$ ***is parallel to xz-plane.*** **(c)** ***Plane*** $cz + d = 0$ ***is parallel to xy-plane.***

**Figure 10.34**

## Distance Between a Point and a Plane

The distance $D$ between a point $Q$ and a plane is the length of the shortest line segment connecting $Q$ to the plane, as shown in Figure 10.35. If $P$ is *any* point in the plane, you can find this distance by projecting the vector $\overrightarrow{PQ}$ onto the normal vector $\mathbf{n}$. The length of this projection is the desired distance.




$D = \|\text{proj}_{\mathbf{n}} \overrightarrow{PQ}\|$

**Figure 10.35**

### Distance Between a Point and a Plane

The **distance between a plane and a point $Q$** (not in the plane) is

$$D = \|\text{proj}_{\mathbf{n}} \overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

where $P$ is a point in the plane and $\mathbf{n}$ is normal to the plane.

To find a point in the plane given by $ax + by + cz + d = 0$, where $a \neq 0$, let $y = 0$ and $z = 0$. Then, from the equation $ax + d = 0$, you can conclude that the point $(-d/a, 0, 0)$ lies in the plane.

### Example 5 Finding the Distance Between a Point and a Plane

Find the distance between the point $Q(1, 5, -4)$ and the plane $3x - y + 2z = 6$.

#### Solution

You know that $\mathbf{n} = \langle 3, -1, 2 \rangle$ is normal to the given plane. To find a point in the plane, let $y = 0$ and $z = 0$, and obtain the point $P(2, 0, 0)$. The vector from $P$ to $Q$ is

$$\begin{aligned} \overrightarrow{PQ} &= \langle 1 - 2, 5 - 0, -4 - 0 \rangle \\ &= \langle -1, 5, -4 \rangle. \end{aligned}$$

The formula for the distance between a point and a plane produces

$$\begin{aligned} D &= \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} \\ &= \frac{|\langle -1, 5, -4 \rangle \cdot \langle 3, -1, 2 \rangle|}{\sqrt{9 + 1 + 4}} \\ &= \frac{|-3 - 5 - 8|}{\sqrt{14}} \\ &= \frac{16}{\sqrt{14}}. \end{aligned}$$

CHECKPOINT Now try Exercise 57.

The choice of the point $P$ in Example 5 is arbitrary. Try choosing a different point to verify that you obtain the same distance.

# 10.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

## *Vocabulary Check*

**Fill in the blanks.**

1. The _______ vector for a line $L$ is given by $\mathbf{v} =$ _______ .
2. The _______ of a line in space are given by $x = x_1 + at$, $y = y_1 + bt$, and $z = z_1 + ct$.
3. If the direction numbers $a$, $b$, and $c$ of the vector $\mathbf{v} = \langle a, b, c\rangle$ are all nonzero, you can eliminate the parameter to obtain the _______ of a line.
4. A vector that is perpendicular to a plane is called _______ .
5. The standard form of the equation of a plane is given by _______ .

---

**In Exercises 1–6, find (a) a set of parametric equations and (b) if possible, a set of symmetric equations for the line through the point and parallel to the specified vector or line. For each line, write the direction numbers as integers. (There are many correct answers.)**

| | *Point* | *Parallel to* |
|---|---|---|
| **1.** | $(0, 0, 0)$ | $\mathbf{v} = \langle 1, 2, 3\rangle$ |
| **2.** | $(3, -5, 1)$ | $\mathbf{v} = \langle 3, -7, -10\rangle$ |
| **3.** | $(-4, 1, 0)$ | $\mathbf{v} = \frac{1}{2}\mathbf{i} + \frac{4}{3}\mathbf{j} - \mathbf{k}$ |
| **4.** | $(5, 0, 10)$ | $\mathbf{v} = 4\mathbf{i} + 3\mathbf{k}$ |
| **5.** | $(2, -3, 5)$ | $x = 5 + 2t, y = 7 - 3t,$ $z = -2 + t$ |
| **6.** | $(1, 0, 1)$ | $x = 3 + 3t, y = 5 - 2t,$ $z = -7 + t$ |

**In Exercises 7–14, find (a) a set of parametric equations and (b) if possible, a set of symmetric equations of the line that passes through the given points. For each line, write the direction numbers as integers. (There are many correct answers.)**

**7.** $(2, 0, 2), (1, 4, -3)$
**8.** $(2, 3, 0), (10, 8, 12)$
**9.** $(-3, 8, 15), (1, -2, 16)$
**10.** $(2, 3, -1), (1, -5, 3)$
**11.** $(3, 1, 2), (-1, 1, 5)$
**12.** $(2, -1, 5), (2, 1, -3)$
**13.** $\left(-\frac{1}{2}, 2, \frac{1}{2}\right), \left(1, -\frac{1}{2}, 0\right)$
**14.** $\left(-\frac{3}{2}, \frac{3}{2}, 2\right), (3, -5, -4)$

**In Exercises 15 and 16, sketch a graph of the line.**

**15.** $x = 2t, y = 2 + t,$ $z = 1 + \frac{1}{2}t$
**16.** $x = 5 - 2t, y = 1 + t,$ $z = 5 - \frac{1}{2}t$

**In Exercises 17–22, find the general form of the equation of the plane passing through the point and normal to the specified vector or line.**

| | *Point* | *Perpendicular to* |
|---|---|---|
| **17.** | $(2, 1, 2)$ | $\mathbf{n} = \mathbf{i}$ |
| **18.** | $(1, 0, -3)$ | $\mathbf{n} = \mathbf{k}$ |
| **19.** | $(5, 6, 3)$ | $\mathbf{n} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ |
| **20.** | $(0, 0, 0)$ | $\mathbf{n} = -3\mathbf{j} + 5\mathbf{k}$ |
| **21.** | $(2, 0, 0)$ | $x = 3 - t, y = 2 - 2t, z = 4 + t$ |
| **22.** | $(0, 0, 6)$ | $x = 1 - t, y = 2 + t, z = 4 - 2t$ |

**In Exercises 23–26, find the general form of the equation of the plane passing through the three points.**

**23.** $(0, 0, 0), (1, 2, 3), (-2, 3, 3)$
**24.** $(4, -1, 3), (2, 5, 1), (-1, 2, 1)$
**25.** $(2, 3, -2), (3, 4, 2), (1, -1, 0)$
**26.** $(5, -1, 4), (1, -1, 2), (2, 1, -3)$

**In Exercises 27–32, find the general form of the equation of the plane with the given characteristics.**

**27.** Passes through $(2, 5, 3)$ and is parallel to $xz$-plane
**28.** Passes through $(1, 2, 3)$ and is parallel to $yz$-plane
**29.** Passes through $(0, 2, 4)$ and $(-1, -2, 0)$ and is perpendicular to $yz$-plane
**30.** Passes through $(1, -2, 4)$ and $(4, 0, -1)$ and is perpendicular to $xz$-plane
**31.** Passes through $(2, 2, 1)$ and $(-1, 1, -1)$ and is perpendicular to $2x - 3y + z = 3$
**32.** Passes through $(1, 2, 0)$ and $(-1, -1, 2)$ and is perpendicular to $2x - 3y + z = 6$

**In Exercises 33–40, find a set of parametric equations of the line. (There are many correct answers.)**

**33.** Passes through $(2, 3, 4)$ and is parallel to $xz$-plane and the $yz$-plane
**34.** Passes through $(-4, 5, 2)$ and is parallel to $xy$-plane and the $yz$-plane

**35.** Passes through $(2, 3, 4)$ and is perpendicular to the plane given by $3x + 2y - z = 6$

**36.** Passes through $(-4, 5, 2)$ and is perpendicular to the plane given by $-x + 2y + z = 5$

**37.** Passes through $(5, -3, -4)$ and is parallel to $\mathbf{v} = \langle 2, -1, 3\rangle$

**38.** Passes through $(-1, 4, -3)$ and is parallel to $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$

**39.** Passes through $(2, 1, 2)$ and is parallel to $x = -t$, $y = 1 + t$, $z = -2 + t$

**40.** Passes through $(-6, 0, 8)$ and is parallel to $x = 5 - 2t$, $y = -4 + 2t$, $z = 0$

**In Exercises 41–44, determine whether the planes are parallel, orthogonal, or neither. If they are neither parallel nor orthogonal, find the angle of intersection.**

**41.** $5x - 3y + z = 4$
$x + 4y + 7z = 1$

**42.** $3x + y - 4z = 3$
$-9x - 3y + 12z = 4$

**43.** $2x - z = 1$
$4x + y + 8z = 10$

**44.** $x - 5y - z = 1$
$5x - 25y - 5z = -3$

**In Exercises 45–48, (a) find the angle between the two planes and (b) find parametric equations of their line of intersection.**

**45.** $3x - 4y + 5z = 6$
$x + y - z = 2$

**46.** $x - 3y + z = -2$
$2x + 5z + 3 = 0$

**47.** $x + y - z = 0$
$2x - 5y - z = 1$

**48.** $2x + 4y - 2z = 1$
$-3x - 6y + 3z = 10$

**In Exercises 49–54, plot the intercepts and sketch a graph of the plane.**

**49.** $x + 2y + 3z = 6$

**50.** $2x - y + 4z = 4$

**51.** $x + 2y = 4$

**52.** $y + z = 5$

**53.** $3x + 2y - z = 6$

**54.** $x - 3z = 6$

**In Exercises 55–58, find the distance between the point and the plane.**

**55.** $(0, 0, 0)$
$8x - 4y + z = 8$

**56.** $(3, 2, 1)$
$x - y + 2z = 4$

**57.** $(4, -2, -2)$
$2x - y + z = 4$

**58.** $(-1, 2, 5)$
$2x + 3y + z = 12$

**59.** ***Machine Design*** A tractor fuel tank has the shape and dimensions shown in the figure. In fabricating the tank, it is necessary to know the angle between two adjacent sides. Find this angle.

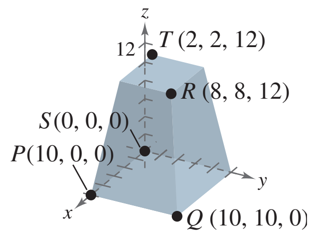



Figure for 59

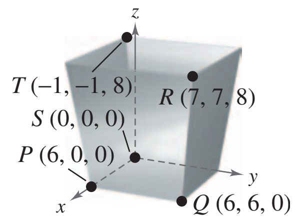



Figure for 60

**60.** ***Mechanical Design*** A chute at the top of a grain elevator of a combine funnels the grain into a bin, as shown in the figure. Find the angle between two adjacent sides.

## Synthesis

***True or False?*** **In Exercises 61 and 62, determine whether the statement is true or false. Justify your answer.**

**61.** Every two lines in space are either intersecting or parallel.

**62.** Two nonparallel planes in space will always intersect.

**63.** The direction numbers of two distinct lines in space are 10, $-18$, 20, and $-15$, 27, $-30$. What is the relationship between the lines? Explain.

**64.** ***Exploration***

(a) Describe and find an equation for the surface generated by all points $(x, y, z)$ that are two units from the point $(4, -1, 1)$.

(b) Describe and find an equation for the surface generated by all points $(x, y, z)$ that are two units from the plane $4x - 3y + z = 10$.

## Skills Review

**In Exercises 65–68, convert the polar equation to rectangular form.**

**65.** $r = 10$

**66.** $\theta = \dfrac{3\pi}{4}$

**67.** $r = 3\cos\theta$

**68.** $r = \dfrac{1}{2 - \cos\theta}$

**In Exercises 69–72, convert the rectangular equation to polar form.**

**69.** $x^2 + y^2 = 49$

**70.** $x^2 + y^2 - 4x = 0$

**71.** $y = 5$

**72.** $2x - y + 1 = 0$

# What Did You Learn?

## Key Terms

solid analytic geometry, *p. 742*
three-dimensional coordinate system, *p. 742*
*xy*-plane, *xz*-plane, and *yz*-plane, *p. 742*
octants, *p. 742*
left-handed or right-handed orientation, *p. 742*
surface in space, *p. 745*
trace, *p. 746*
zero vector, *p. 750*
standard unit vector notation, *p. 750*
component form, *p. 750*
orthogonal, *p. 751*
parallel, *p. 752*
collinear, *p. 753*
triple scalar product, *p. 761*
direction vector, *p. 764*
direction numbers, *p. 764*
parametric equations, *p. 764*
symmetric equations, *p. 764*
general form of the equation of a plane, *p. 766*
angle between two planes, *p. 767*

## Key Concepts

### 10.1 ■ Find the distance between two points in space

The distance between the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ given by the Distance Formula in Space is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

### 10.1 ■ Find the midpoint of a line segment joining two points in space

The midpoint of the line segment joining the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ given by the Midpoint Formula in Space is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_2 + z_2}{2}\right)$$

### 10.1 ■ Write the standard equation of a sphere

The standard equation of a sphere with center $(h, k, j)$ and radius $r$ is given by

$$(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2.$$

### 10.2 ■ Use vectors in space

1. Two vectors are equal if and only if their corresponding components are equal.
2. The magnitude (or length) of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is
$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$
3. A unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$ is
$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}, \quad \mathbf{v} \neq 0.$$
4. The sum of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is
$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle.$$
5. The scalar multiple of the real number $c$ and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is $c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$.
6. The dot product of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$.

### 10.2 ■ Find the angle between two vectors

If $\theta$ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

### 10.3 ■ Use the cross product of two vectors

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in space. The cross product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space and let $c$ be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

### 10.4 ■ Write parametric equations of a line in space

A line $L$ parallel to nonzero the vector $\mathbf{v} = \langle a, b, c \rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the parametric equations $x = x_1 + at$, $y = y_1 + bt$, and $z = z_1 + ct$.

### 10.4 ■ Write the standard equation of a plane in space

The plane containing the point $(x_1, y_1, z_1)$ and having nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$ can be represented by the standard form of the equation of a plane

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

## Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**10.1 In Exercises 1 and 2, plot each point in the same three-dimensional coordinate system.**

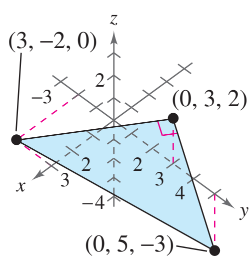
1. (a) $(5, -1, 2)$ (b) $(-3, 3, 0)$
2. (a) $(2, 4, -3)$ (b) $(0, 0, 5)$
3. Find the coordinates of the point in the $xy$-plane four units to the right of the $xz$-plane and five units behind the $yz$-plane.
4. Find the coordinates of the point located on the $y$-axis and seven units to the left of the $xz$-plane.

**In Exercises 5 and 6, find the distance between the points.**

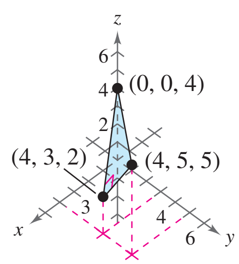
5. $(4, 0, 7), (5, 2, 1)$
6. $(2, 3, -4), (-1, -3, 0)$

**In Exercises 7 and 8, find the lengths of the sides of the right triangle. Show that these lengths satisfy the Pythagorean Theorem.**

7.




8.




**In Exercises 9–12, find the midpoint of the line segment joining the points.**

9. $(-2, 3, 2), (2, -5, -2)$
10. $(7, 1, -4), (1, -1, 2)$
11. $(10, 6, -12), (-8, -2, -6)$
12. $(-5, -3, 1), (-7, -9, -5)$

**In Exercises 13–16, find the standard form of the equation of the sphere with the given characteristics.**

13. Center: $(2, 3, 5)$; radius: 1
14. Center: $(3, -2, 4)$; radius: 4
15. Center: $(1, 5, 2)$; diameter: 12
16. Center: $(0, 4, -1)$; diameter: 15

**In Exercises 17 and 18, find the center and radius of the sphere.**

17. $x^2 + y^2 + z^2 - 4x - 6y + 4 = 0$
18. $x^2 + y^2 + z^2 - 10x + 6y - 4z + 34 = 0$

**In Exercises 19 and 20, sketch the graph of the equation and sketch the specified trace.**

19. $x^2 + (y - 3)^2 + z^2 = 16$
 (a) $xz$-trace (b) $yz$-trace
20. $(x + 2)^2 + (y - 1)^2 + z^2 = 9$
 (a) $xy$-trace (b) $yz$-trace

**10.2 In Exercises 21–24, (a) write the component form of the vector v, (b) find the magnitude of v, and (c) find a unit vector in the direction of v.**

| | *Initial point* | *Terminal point* |
|---|---|---|
| 21. | $(2, -1, 4)$ | $(3, 3, 0)$ |
| 22. | $(2, -1, 2)$ | $(-3, 2, 3)$ |
| 23. | $(7, -4, 3)$ | $(-3, 2, 10)$ |
| 24. | $(0, 3, -1)$ | $(5, -8, 6)$ |

**In Exercises 25–28, find the dot product of u and v.**

25. $\mathbf{u} = \langle -1, 4, 3\rangle$, $\mathbf{v} = \langle 0, -6, 5\rangle$
26. $\mathbf{u} = \langle 8, -4, 2\rangle$, $\mathbf{v} = \langle 2, 5, 2\rangle$
27. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$
28. $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

**In Exercises 29–32, find the angle $\theta$ between the vectors.**

29. $\mathbf{u} = \langle 4, -1, 5\rangle$, $\mathbf{v} = \langle 3, 2, -2\rangle$
30. $\mathbf{u} = \langle 10, -5, 15\rangle$, $\mathbf{v} = \langle -2, 1, -3\rangle$
31. $\mathbf{u} = \langle 2\sqrt{2}, -4, 4\rangle$, $\mathbf{v} = \langle -\sqrt{2}, 1, 2\rangle$
32. $\mathbf{u} = \langle 3, 1, -1\rangle$, $\mathbf{v} = \langle 4, 5, 2\rangle$

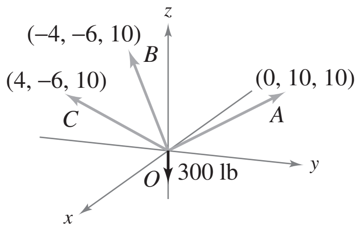
**In Exercises 33–36, determine whether u and v are orthogonal, parallel, or neither.**

33. $\mathbf{u} = \langle 7, -2, 3\rangle$, $\mathbf{v} = \langle -1, 4, 5\rangle$
34. $\mathbf{u} = \langle -4, 3, -6\rangle$, $\mathbf{v} = \langle 16, -12, 24\rangle$
35. $\mathbf{u} = \langle 39, -12, 21\rangle$, $\mathbf{v} = \langle -26, 8, -14\rangle$
36. $\mathbf{u} = \langle 8, 5, -8\rangle$, $\mathbf{v} = \left\langle -2, 4, \frac{1}{2}\right\rangle$

**In Exercises 37–40, use vectors to determine whether the points are collinear.**

37. $(5, 2, 0), (2, 6, 1), (2, 4, 7)$
38. $(6, 3, -1), (5, 8, 3), (7, -2, -5)$
39. $(3, 4, -1), (-1, 6, 9), (5, 3, -6)$
40. $(5, -4. 7), (8, -5, 5), (11, 6, 3)$

**41.** ***Tension*** A load of 300 pounds is supported by three cables, as shown in the figure. Find the tension in each of the supporting cables.




**42.** ***Tension*** Determine the tension in each of the supporting cables in Exercise 41 if the load is 200 pounds.

**10.3 In Exercises 43 and 44, find u × v.**

**43.** $\mathbf{u} = \langle -2, 8, 2\rangle$, $\mathbf{v} = \langle 1, 1, -1\rangle$

**44.** $\mathbf{u} = \langle 10, 15, 5\rangle$, $\mathbf{v} = \langle 5, -3, 0\rangle$

**In Exercises 45 and 46, find a unit vector orthogonal to u and v.**

**45.** $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{v} = 10\mathbf{i} - 15\mathbf{j} + 2\mathbf{k}$

**46.** $\mathbf{u} = 4\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 12\mathbf{k}$

**In Exercises 47 and 48, verify that the points are the vertices of a parallelogram and find its area.**

**47.** $A(2, -1, 1)$, $B(5, 1, 4)$, $C(0, 1, 1)$, $D(3, 3, 4)$

**48.** $A(0, 4, 0)$, $B(1, 4, 1)$, $C(0, 6, 0)$, $D(1, 6, 1)$

**In Exercises 49 and 50, find the volume of the parallelepiped with the given vertices.**

**49.** $A(0, 0, 0)$, $B(3, 0, 0)$, $C(0, 5, 1)$, $D(3, 5, 1)$, $E(2, 0, 5)$, $F(5, 0, 5)$, $G(2, 5, 6)$, $H(5, 5, 6)$

**50.** $A(0, 0, 0)$, $B(2, 0, 0)$, $C(2, 4, 0)$, $D(0, 4, 0)$, $E(0, 0, 6)$, $F(2, 0, 6)$, $G(2, 4, 6)$, $H(0, 4, 6)$

**10.4 In Exercises 51 and 52, find sets of (a) parametric equations and (b) symmetric equations for the line that passes through the given points. For each line, write the direction numbers as intergers. (There are many correct answers.)**

**51.** $(3, 0, 2)$, $(9, 11, 6)$

**52.** $(-1, 4, 3)$, $(8, 10, 5)$

**In Exercises 53–56, find a set of (a) parametric equations and (b) symmetric equations for the specified line. (There are many correct answers.)**

**53.** Passes through $(-1, 3, 5)$ and $(3, 6, -1)$

**54.** Passes through $(0, -10, 3)$ and $(5, 10, 0)$

**55.** Passes through $(0, 0, 0)$ and is parallel to $\mathbf{v} = \left\langle -2, \frac{5}{2}, 1\right\rangle$

**56.** Passes through $(3, 2, 1)$ and is parallel to $x = y = z$

**In Exercises 57–60, find the general form of the equation of the plane with the given characteristics.**

**57.** Passes through $(0, 0, 0)$, $(5, 0, 2)$, and $(2, 3, 8)$

**58.** Passes through $(-1, 3, 4)$, $(4, -2, 2)$, and $(2, 8, 6)$

**59.** Passes through $(5, 3, 2)$ and is parallel to the $xy$-plane

**60.** Passes through $(0, 0, 6)$ and is perpendicular to $x = 1 - t$, $y = 2 + t$, and $z = 4 - 2t$

**In Exercises 61–64, plot the intercepts and sketch a graph of the plane.**

**61.** $3x - 2y + 3z = 6$

**62.** $5x - y - 5z = 5$

**63.** $2x - 3z = 6$

**64.** $4y - 3z = 12$

**In Exercises 65–68, find the distance between the point and the plane.**

**65.** $(2, 3, 10)$, $2x - 20y + 6z = 6$

**66.** $(1, 2, 3)$, $2x - y + z = 4$

**67.** $(0, 0, 0)$, $x - 10y + 3z = 2$

**68.** $(0, 0, 0)$, $2x + 3y + z = 12$

## Synthesis

***True or False?*** **In Exercises 69 and 70, determine whether the statement is true or false. Justify your answer.**

**69.** The cross product is commutative.

**70.** The triple scalar product of three vectors in space is a scalar.

**In Exercises 71–74, let $\mathbf{u} = \langle 3, -2, 1\rangle$, $\mathbf{v} = \langle 2, -4, -3\rangle$, and $\mathbf{w} = \langle -1, 2, 2\rangle$.**

**71.** Show that $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.

**72.** Show that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

**73.** Show that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

**74.** Show that $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.

**75.** ***Writing*** Define the cross product of vectors **u** and **v**.

**76.** ***Writing*** State the geometric properties of the cross product.

**77.** ***Writing*** If the magnitudes of two vectors are doubled, how will the magnitude of the cross product of vectors change?

**78.** ***Writing*** The vertices of a triangle in space are $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, and $(x_3, y_3, z_3)$. Explain how to find a vector perpendicular to the triangle.

# 10 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.**

**1.** Plot each point in the same three-dimensional coordinate system.

(a) $(5, -2, 3)$ (b) $(-2, -2, 3)$ (c) $(-1, 2, 1)$

**In Exercises 2–4, use the points $A(8, -2, 5)$, $B(6, 4, -1)$, and $C(-4, 3, 0)$ to solve the problem.**

**2.** Consider the triangle with vertices $A$, $B$, and $C$. Is it a right triangle? Explain.

**3.** Find the coordinates of the midpoint of the line segment joining points $A$ and $B$.

**4.** Find the standard form of the equation of the sphere for which $A$ and $B$ are the endpoints of a diameter. Sketch the sphere and its $xz$-trace.

**In Exercises 5 and 6, find the component form and the magnitude of the vector v.**

**5.** Initial point: $(2, -1, 3)$
Terminal point: $(4, 4, -7)$

**6.** Initial point: $(6, 2, 0)$
Terminal point: $(3, -3, 8)$

**In Exercises 7–10, let u and v be the vectors from $A(8, -2, 5)$ to $B(6, 4, -1)$ and from $A$ to $C(-4, 3, 0)$ respectively.**

**7.** Write $\mathbf{u}$ and $\mathbf{v}$ in component form.

**8.** Find (a) $\|\mathbf{v}\|$, (b) $\mathbf{u} \cdot \mathbf{v}$, and (c) $\mathbf{u} \times \mathbf{v}$.

**9.** Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

**10.** Find a set of (a) parametric equations and (b) symmetric equations for the line through points $A$ and $B$.

**In Exercises 11 and 12, determine whether u and v are orthogonal, parallel, or neither.**

**11.** $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$
$\mathbf{v} = \mathbf{j} + 6\mathbf{k}$

**12.** $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
$\mathbf{v} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$

**13.** Verify that the points $A(2. -3, 1)$, $B(6, 5, -1)$, $C(3, -6, 4)$, and $D(7, 2, 2)$ are the vertices of a parallelogram, and find its area.

**14.** Find the general form of the equation of the plane passing through the points $(-3, -4, 2)$, $(-3, 4, 1)$, and $(1, 1, -2)$.

**15.** Find the volume of the parallelepiped at the right with the given vertices.
$A(0, 0, 5)$, $B(0, 10, 5)$, $C(4, 10, 5)$, $D(4, 0, 5)$,
$E(0, 1, 0)$, $F(0, 11, 0)$, $G(4, 11, 0)$, $H(4, 1, 0)$

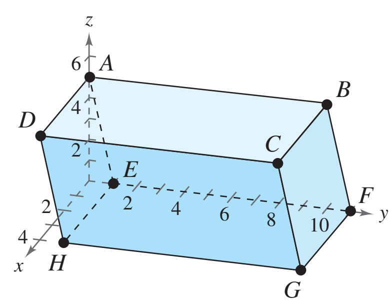



Figure for 15

**In Exercises 16 and 17, label the intercepts and sketch a graph of the plane.**

**16.** $2x + 3y + 4z = 12$

**17.** $5x - y - 2z = 10$

**18.** Find the distance between the point $(2, -1, 6)$ and the plane $3x - 2y + z = 6$.

# Proofs in Mathematics

**Algebraic Properties of the Cross Product** (p. 758)

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space and let $c$ be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

## Proof

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, and let $c$ be a scalar.

**1.** $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

$\mathbf{v} \times \mathbf{u} = (v_2u_3 - v_3u_2)\mathbf{i} - (v_1u_3 - v_3u_1)\mathbf{j} + (v_1u_2 - v_2u_1)\mathbf{k}$

So, this implies $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

**2.**
$$\begin{aligned}\mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= [u_2(v_3 + w_3) - u_3(v_2 + w_2)]\mathbf{i} - [u_1(v_3 + w_3) - \\ &\qquad u_3(v_1 + w_1)]\mathbf{j} + [u_1(v_2 + w_2) - u_2(v_1 + w_1)]\mathbf{k} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} + \\ &\qquad (u_2w_3 - u_3w_2)\mathbf{i} - (u_1w_3 - u_3w_1)\mathbf{j} + (u_1w_2 - u_2w_1)\mathbf{k} \\ &= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})\end{aligned}$$

**3.**
$$\begin{aligned}(c\mathbf{u}) \times \mathbf{v} &= (cu_2v_3 - cu_3v_2)\mathbf{i} - (cu_1v_3 - cu_3v_1)\mathbf{j} + (cu_1v_2 - cu_2v_1)\mathbf{k} \\ &= c[(u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}] = c(\mathbf{u} \times \mathbf{v})\end{aligned}$$

**4.**
$$\begin{aligned}\mathbf{u} \times \mathbf{0} &= (u_2 \cdot 0 - u_3 \cdot 0)\mathbf{i} - (u_1 \cdot 0 - u_3 \cdot 0)\mathbf{j} + (u_1 \cdot 0 - u_2 \cdot 0)\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}\end{aligned}$$
$$\begin{aligned}\mathbf{0} \times \mathbf{u} &= (0 \cdot u_3 - 0 \cdot u_2)\mathbf{i} - (0 \cdot u_3 - 0 \cdot u_1)\mathbf{j} + (0 \cdot u_2 - 0 \cdot u_1)\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}\end{aligned}$$

So, this implies $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$.

**5.** $\mathbf{u} \times \mathbf{u} = (u_2u_3 - u_3u_2)\mathbf{i} - (u_1u_3 - u_3u_1)\mathbf{j} + (u_1u_2 - u_2u_1)\mathbf{k} = \mathbf{0}$

**6.** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ and $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= u_1(v_2w_3 - w_2v_3) - u_2(v_1w_3 - w_1v_3) + u_3(v_1w_2 - w_1v_2) \\ &= u_1v_2w_3 - u_1w_2v_3 - u_2v_1w_3 + u_2w_1v_3 + u_3v_1w_2 - u_3w_1v_2 \\ &= u_2w_1v_3 - u_3w_1v_2 - u_1w_2v_3 + u_3v_1w_2 + u_1v_2w_3 - u_2v_1w_3 \\ &= w_1(u_2v_3 - v_2u_3) - w_2(u_1v_3 - v_1u_3) + w_3(u_1v_2 - v_1u_2) \\ &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}\end{aligned}$$

### Notation for Dot and Cross Products

The notation for the dot products and cross products of vectors was first introduced by the American physicist Josiah Willard Gibbs (1839–1903). In the early 1880s, Gibbs built a system to represent physical quantities called *vector analysis*. The system was a departure from William Hamilton's theory of quaternions.

**Geometric Properties of the Cross Product** (p. 759)

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let $\theta$ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

**1.** $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

**2.** $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$

**3.** $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.

**4.** $\|\mathbf{u} \times \mathbf{v}\| =$ area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

## Proof

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$.

**1.**
$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \\
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} &= (u_2v_3 - u_3v_2)u_1 - (u_1v_3 - u_3v_1)u_2 + (u_1v_2 - u_2v_1)u_3 \\
&= u_1u_2v_3 - u_1u_3v_2 - u_1u_2v_3 + u_2u_3v_1 + u_1u_3v_2 - u_2u_3v_1 = 0 \\
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} &= (u_2v_3 - u_3v_2)v_1 - (u_1v_3 - u_3v_1)v_2 + (u_1v_2 - u_2v_1)v_3 \\
&= u_2v_1v_3 - u_3v_1v_2 - u_1v_2v_3 + u_3v_1v_2 + u_1v_2v_3 - u_2v_1v_3 = 0
\end{aligned}$$

Because two vectors are orthogonal if their dot product is zero, it follows that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

**2.** Note that $\cos \theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$. Therefore,

$$\begin{aligned}
\|\mathbf{u}\| \|\mathbf{v}\| \sin \theta &= \|\mathbf{u}\| \|\mathbf{v}\| \sqrt{1 - \cos^2 \theta} \\
&= \|\mathbf{u}\| \|\mathbf{v}\| \sqrt{1 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2}} \\
&= \sqrt{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \\
&= \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2} \\
&= \sqrt{(u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2} = \|\mathbf{u} \times \mathbf{v}\|.
\end{aligned}$$

**3.** If $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other, then $\mathbf{u} = c\mathbf{v}$ for some scalar $c$.

$$\mathbf{u} \times \mathbf{v} = (c\mathbf{v}) \times \mathbf{v} = c(\mathbf{v} \times \mathbf{v}) = c(\mathbf{0}) = \mathbf{0}$$

If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\|\mathbf{u}\| \|\mathbf{v}\| \sin \theta = 0$. (Assume $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$.) So, $\sin \theta = 0$, and $\theta = 0$ or $\theta = \pi$. In either case, because $\theta$ is the angle between the vectors, $\mathbf{u}$ and $\mathbf{v}$ are parallel. Therefore, $\mathbf{u} = c\mathbf{v}$ for some scalar $c$.

**4.** The figure at the left is a parallelogram having $\mathbf{v}$ and $\mathbf{u}$ as adjacent sides. Because the height of the parallelogram is $\|\mathbf{v}\| \sin \theta$, the area is

$$\begin{aligned}
\text{Area} &= (\text{base})(\text{height}) \\
&= \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta \\
&= \|\mathbf{u} \times \mathbf{v}\|.
\end{aligned}$$